104114 Lecture Notes

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1 Euclidean Geometry

An *Euclidean Space* is an n-dimensional linear space V over the field $\mathbb{R}$ of real numbers, together with a inner product $\langle\cdot,\cdot\rangle$. Recall that this means that $\langle\cdot,\cdot\rangle$ is symmetric bilinear form such that $\langle u,u\rangle \geq 0$, where equality implies that $u=0$. The form $\langle\cdot,\cdot\rangle$ induces a norm $\|\cdot\|$ on V given by $\|u\| = \sqrt{\langle u,u\rangle}$. Any n-dimensional Euclidean space V is isomorphic to $\mathbb{R}^n$ with its standard inner product defined on $u=(a_1,\ldots,a_n), v=(b_1,\ldots,b_n)$ by $\langle u,v\rangle = \sum_{i=1}^n a_i b_i$. The angle α between two nonzero vectors $u,v\in V$ is given by $\alpha = \arccos\frac{\langle u,v\rangle}{\|u\|\cdot\|v\|}$. An *affine combination* of $u_0,\ldots,u_k \in V$ is a vector of the form $\sum_{i=0}^k \lambda_i u_i$ where $\sum_{i=0}^k \lambda_i = 1$. The *affine span* of $A \subset V$ is

$$\mathrm{aff}\,(A) = \left\{ \sum_{i=0}^k \lambda_i u_i : u_i \in A, \sum_{i=0}^k \lambda_i = 1 \right\}.$$

The vectors $u_0,\ldots,u_k \in V$ are *affinely independent* if $\sum_{i=0}^k \lambda_i u_i = 0$ together with $\sum_{i=0}^k \lambda_i = 0$, imply that $\lambda_i = 0$ for all $0\leq i\leq k$. An equivalent condition (check!) is that $u_j \notin \mathrm{aff}\,(\{u_i\}_{i\neq j})$ for all $0\leq j\leq k$. A subset $F\subset V$ is a *flat* if $\mathrm{aff}\,(F) = F$. Check that F is a flat iff $F = v+U$, for some $v\in V$ a linear subspace U of V. The subspace U is uniquely determined by F (check!), and is called the *direction* of F. We define $\dim F = \dim U$.

A set $K\subset\mathbb{R}^n$ is *convex* if for any $u,v\in K$, the segment $[u,v] := \{tu+(1-t)v : 0\leq t\leq 1\}$ is contained in K. The standard k-simplex in $\mathbb{R}^{k+1}$ is the set

$$\Delta_k = \{\lambda = (\lambda_0,\ldots,\lambda_k)\in\mathbb{R}^{k+1} : \lambda_i \geq 0, \sum_{i=0}^k \lambda_i = 1\}.$$

A convex combination of $u_0,\ldots,u_k$ is a vector of the form $\sum_{i=0}^k \lambda_i u_i$ where $(\lambda_0,\ldots,\lambda_k)\in\Delta_k\}$. The *convex hull* of a set $A\in\mathbb{R}^n$ is

$$\mathrm{conv}\,(A) = \left\{ \sum_{i=0}^k \lambda_i u_i : u_0,\ldots,u_k \in A, (\lambda_0,\ldots,\lambda_k)\in\Delta_k \right\}.$$

$\mathrm{conv}\,(A)$ is the inclusion-wise minimal convex subset of $\mathbb{R}^n$ that contains A (check!).

In the following sections we will recall some classical results from plane Euclidean geometry, and discuss their higher dimensional counterparts.

1.1 Ceva's Theorem in n-Space

A *Cevian* in a triangle is a segment that connects a vertex to a point in the opposite edge. In plane geometry we encounter several results that assert that certain three Cevians from the three vertices are concurrent (i.e. intersect in a point). For example:

Theorem 1.1. *In a triangle* $\Delta = ABC$
(i) The three medians are concurrent.
(ii) The three altitudes are concurrent.
(iii) The three angle bisectors are concurrent.

It turns out that Theorem 1.1 and similar results can be proved as a consequence of the following

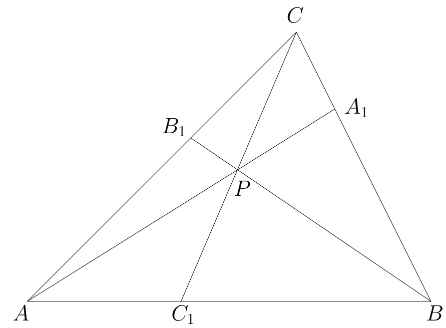

Figure 1: Three collinear Cevians

Theorem 1.2 (Ceva). *Let* ΔABC *be a triangle in the plane and consider three points* $C_1 \in \overline{AB}$ *,*$A_1 \in \overline{BC}$*, and* $B_1 \in \overline{CA}$*. Then the segments* $\overline{AA_1}$*,* $\overline{BB_1}$*,* $\overline{CC_1}$ *intersect in a point* P *iff*

$$\frac{|AC_1| \cdot |BA_1| \cdot |CB_1|}{|BC_1| \cdot |CA_1| \cdot |AB_1|} = 1. \tag{1}$$

Proof of Theorem 1.1. Let $\alpha = \angle BAC$, $\beta = \angle ABC$ and $\gamma = \angle ACB$. Let $a = |BC|$, $b = |AC|$ and $c = |AB|$.
(i) Let AA_1, BB_1, CC_1 be the three medians of Δ. Then $|AC_1| = |BC_1| = \frac{c}{2}$, $|BA_1| = |CA_1| = \frac{a}{2}$ and $|CB_1| = |AB_1| = \frac{b}{2}$. It follows that (1) is satisfied and hence AA_1, BB_1, CC_1 are concurrent.

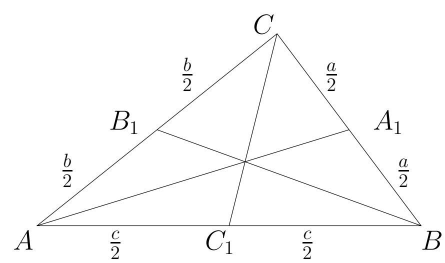

Figure 2: Medians

(ii) Let AA_1, BB_1, CC_1 be the three altitudes of Δ. Then $|BA_1| = c\cos\beta$, $|CB_1| = a\cos\gamma$ and $|AC_1| = b\cos\alpha$. Similarly $|CA_1| = b\cos\gamma$, $|AB_1| = c\cos\alpha$ and $|BC_1| = a\cos\beta$. It follows that

$$\frac{|AC_1| \cdot |BA_1| \cdot |CB_1|}{|BC_1| \cdot |CA_1| \cdot |AB_1|} = \frac{b\cos\alpha \cdot c\cos\beta \cdot a\cos\gamma}{a\cos\beta \cdot b\cos\gamma \cdot c\cos\alpha} = 1,$$

and hence AA_1, BB_1, CC_1 are concurrent.
(iii) Let AA_1, BB_1, CC_1 be the three angle bisectors of Δ. By the sines theorem

$$\frac{|BA_1|}{\sin\frac{\alpha}{2}} = \frac{c}{\sin\angle AA_1B}$$

and

$$\frac{|CA_1|}{\sin\frac{\alpha}{2}} = \frac{b}{\sin\angle AA_1C}.$$

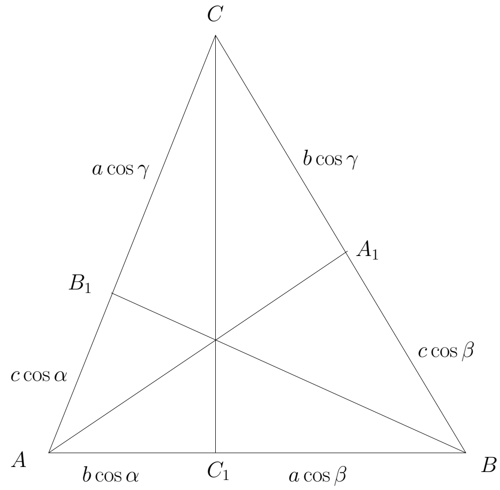

Figure 3: Altitudes

It follows that $\frac{|BA_1|}{|CA_1|} = \frac{c}{b}$. Similarly $\frac{|AC_1|}{|BC_1|} = \frac{b}{a}$ and $\frac{|CB_1|}{|AB_1|} = \frac{a}{c}$. Therefore

$$\frac{|AC_1| \cdot |BA_1| \cdot |CB_1|}{|BC_1| \cdot |CA_1| \cdot |AB_1|} = \frac{b}{a} \cdot \frac{a}{c} \cdot \frac{c}{b} = 1,$$

and hence AA_1, BB_1, CC_1 are concurrent.

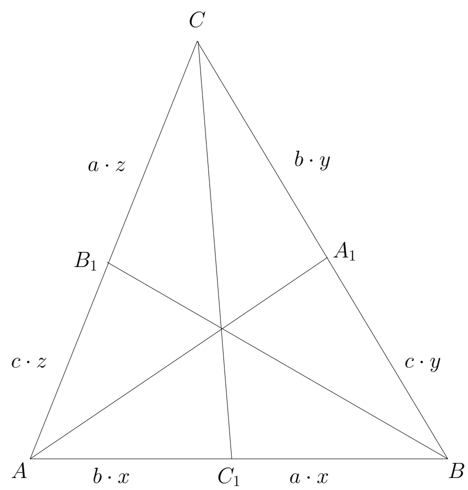

Figure 4: Angle Bisectors

□

For the sequel it will be useful to formulate Ceva's theorem in a different but equivalent form. Fix an arbitrary origin O in $\mathbb{R}^2$, and let $u_A = \overrightarrow{OA}$, $u_B = \overrightarrow{OB}$ and $u_C = \overrightarrow{OC}$. Let $v_A = \overrightarrow{OA_1}$, $v_B = \overrightarrow{OB_1}$ and $v_C = \overrightarrow{OC_1}$.

Claim 1.3. *The Cevians AA_1,BB_1,CC_1 satisfy (1) iff there exist $(\lambda_A, \lambda_B, \lambda_C) \in \Delta_2$ such that*

$$\begin{aligned} v_A &= \frac{1}{1-\lambda_A}(\lambda_B u_B + \lambda_C u_C), \\ v_B &= \frac{1}{1-\lambda_B}(\lambda_A u_A + \lambda_C u_C), \\ v_C &= \frac{1}{1-\lambda_C}(\lambda_A u_A + \lambda_B u_B). \end{aligned} \tag{2}$$

Proof. If (2) holds then

$$\frac{|AC_1|}{|BC_1|} = \frac{\lambda_B}{\lambda_A} \quad , \quad \frac{|BA_1|}{|CA_1|} = \frac{\lambda_C}{\lambda_B} \quad , \quad \frac{|CB_1|}{|AB_1|} = \frac{\lambda_A}{\lambda_C}.$$

Hence

$$\frac{|AC_1| \cdot |BA_1| \cdot |CB_1|}{|BC_1| \cdot |CA_1| \cdot |AB_1|} = \frac{\lambda_B}{\lambda_A} \cdot \frac{\lambda_C}{\lambda_B} \cdot \frac{\lambda_A}{\lambda_C} = 1.$$

Conversely, suppose that (1) holds. Let

$$S = |CB_1| \cdot |BC_1| + |AC_1| \cdot |CB_1| + |AB_1| \cdot |BC_1|$$

and let

$$\lambda_A = \frac{|CB_1| \cdot |BC_1|}{S} \quad , \quad \lambda_B = \frac{|AC_1| \cdot |CB_1|}{S} \quad , \quad \lambda_C = \frac{|AB_1| \cdot |BC_1|}{S}.$$

Then

$$\frac{\lambda_B}{\lambda_A} = \frac{|AC_1| \cdot |CB_1|}{|CB_1| \cdot |BC_1|} = \frac{|AC_1|}{|BC_1|}$$

and

$$\frac{\lambda_C}{\lambda_A} = \frac{|AB_1| \cdot |BC_1|}{|CB_1| \cdot |BC_1|} = \frac{|AB_1|}{|CB_1|}.$$

Moreover, by (1)

$$\frac{\lambda_B}{\lambda_C} = \frac{|AC_1| \cdot |CB_1|}{|AB_1| \cdot |BC_1|} = \frac{|CA_1|}{|BA_1|}.$$

hence (2) holds.

□

In view of Claim 1.3, Ceva's Theorem 1.2 is equivalent to the following

Theorem 1.4. *The Cevians $\overline{AA_1}$, $\overline{BB_1}$, $\overline{CC_1}$ are concurrent iff there exists a point $(\lambda_A, \lambda_B, \lambda_C) \in \Delta_2$ such that (2) holds.*

□

This version of Ceva's theorem admits a straightforward high dimensional extension. As in the planar case, define a Cevian in a simplex $\sigma = \text{conv}\{u_i : 0 \le i \le k\}$ as a segment $[u_j, v_j]$ where $0 \le j \le k$ and $v_j \in \text{conv}\{u_i : 0 \le i \le k, i \ne j\}$.

Theorem 1.5 (Landy)**.** *Let $\sigma = conv\{u_0, \ldots, u_n\}$ be a nondegenerate n-simplex in $\mathbb{R}^n$, and for $0 \le i \le n$ let $v_i \in conv\ \{u_j : 0 \le j \le n, j \ne u\}$. Then*

$$\bigcap_{i=0}^{n} [u_i, v_i] \ne \emptyset \tag{3}$$

iff there exist $\lambda = (\lambda_0, \ldots, \lambda_n) \in \Delta_n$ such that for all $0 \le i \le n$

$$v_i = \frac{1}{1-\lambda_i} \sum_{j \ne i} \lambda_j u_j. \tag{4}$$

Proof. For the direction (4) ⇒ (3), note that if (4) holds then for all $0 \le i \le n$

$$\begin{aligned} p := \sum_{j=0}^{n} \lambda_j u_j &= \lambda_i u_i + (1-\lambda_i) \left(\frac{1}{1-\lambda_i} \sum_{j \ne i} \lambda_j u_j \right) \\ &= \lambda_i u_i + (1-\lambda_i) v_i \in [u_i, v_i]. \end{aligned}$$

For the other direction, suppose $\{[u_i, v_i]\}_{i=0}^n$ are Cevians such that $\{p\} = \cap_{i=0}^n [u_i, v_i]$. For $0 \leq i \leq n$ let $v_i = \sum_{j=0}^n \lambda_{ij} u_j$ where $\lambda_{ij} \geq 0$, $\sum_{j=0}^n \lambda_{ij} = 1$ and $\lambda_{ii} = 0$, and let $p = \theta_i u_i + (1-\theta_i) v_i$. Let $p = \sum_{j=0}^n \mu_j u_j$ where $\mu_i \geq 0$ and $\sum_{i=0}^n \mu_i = 1$. Fix $0 \leq i \leq n$. Then

$$\begin{aligned}\sum_{j=0}^n \mu_j u_j = p &= \theta_i u_i + (1-\theta_i) v_i \\ &= \theta_i u_i + (1-\theta_i) \sum_{j=0}^n \lambda_{ij} u_j \\ &= \theta_i u_i + \sum_{j \neq i} (1-\theta_i) \lambda_{ij} u_j.\end{aligned} \tag{5}$$

It follows that $\theta_i = \mu_i$ and $(1-\theta_i)\lambda_{ij} = \mu_j$ for $j \neq i$. Thus

$$v_i = \sum_{j \neq i} \lambda_{ij} u_j = \frac{1}{1-\mu_i} \sum_{j \neq i} \mu_j u_j.$$

□

The *barycenter* of a simplex $S = \text{conv}\{u_0, \dots, u_k\}$ is $\frac{1}{k+1} \sum_{i=0}^k u_i$. The median from u_i is the Cevian connecting u_i to the barycenter $v_i = \frac{1}{n} \sum_{j \neq i} u_j$ of the face $\text{conv}\{u_j : 0 \leq j \leq k, j \neq i\}$. Taking $(\lambda_0, \dots, \lambda_n) = (\frac{1}{n+1}, \dots, \frac{1}{n+1})$ in Theorem 1.5 we obtain the following

Claim 1.6.

$$\bigcap_{i=0}^n [u_i, v_i] = p = \frac{1}{n+1} \sum_{i=0}^n u_i.$$

An n-simplex $S = \text{conv}\{u_0, \dots, u_n\}$ is *orthocentric* if all altitudes in S intersect in a point, called the *orthocenter*. For $n > 2$, not every n-simplex is orthocentric.

Proposition 1.7. *S is orthocentric iff there exists a constant c such that $(u_i - u_0, u_j - u_0) = c$ for all $i \neq j \in \{1, \dots, n\}$.*

Proof. Suppose S is orthocentric with orthocenter p. then for any distinct nonzero i, j

$$0 = (p - u_0, u_i - u_j) = (p - u_0, (u_i - u_0) - (u_j - u_0)).$$

Therefore $(p - u_0, u_i - u_0) = c$ for all $i \neq 0$. Hence

$$(u_i - u_0, u_j - u_0) = ((p - u_0) - (p - u_i), u_j - u_0) = (p - u_0, u_j - u_0) = c.$$

Conversely, suppose $(u_i - u_0, u_j - u_0) = c$ for all distinct nonzero i, j. Let $\{v_i\}_{i=1}^n$ be a basis dual to the basis $\{u_i - u_0\}_{i=1}^n$, i.e. $(v_i, u_j - u_0) = \delta_{ij}$. We claim that $p = u_0 + c \sum_{i=1}^n v_i$ is the orthocenter of S. Indeed, if $k \neq \ell$ are nonzero, then

$$(p - u_0, u_k - u_\ell) = c(\sum_{i=1}^n v_i, u_k - u_\ell) = c(1-1) = 0.$$

On the other hand

$$\begin{aligned}(p - u_k, u_0 - u_\ell) &= \left((u_0 - u_k) + c \sum_{i=1}^n v_i, u_0 - u_\ell \right) \\ &= (u_0 - u_k, u_0 - u_\ell) - c \left(\sum_{i=1}^n v_i, u_\ell - u_0 \right) \\ &= c - c = 0.\end{aligned}$$

□

1.2 Heron's Formula in n-Space

We start with some remarks on volumes in $\mathbb{R}^n$. Let A be a bounded set in $\mathbb{R}^n$ and let 1_A be the indicator function of A, i.e. $1_A(u) = 1$ if $u \in A$ and $1_A(u) = 0$ otherwise. Assume that the boundary ∂A has measure 0. Then 1_A is Riemann integrable and we define $\mathrm{vol}(A) = \int_{\mathbb{R}^n} 1_A(x)dx$. For $u_1, \ldots, u_n$ let

$$P(u_1, \ldots, u_n) = \left\{ \sum_{i=1}^{n} x_i u_i : 0 \le x_i \le 1 \right\}$$

denote the parallelogram generated by $u_1, \ldots, u_n$. The volume of $P(u_1, \ldots, u_n)$ is given by

$$\mathrm{vol}\left(P(u_1, \ldots, u_n)\right) = |\det(u_1, \ldots, u_n)|. \tag{6}$$

Let $S = \mathrm{conv}\,\{u_0, \ldots, u_n\}$. Then

$$\begin{aligned} \mathrm{vol}(S) &= \frac{1}{n!}\mathrm{vol}\left(P(u_1 - u_0, \ldots, u_n - u_0)\right) \\ &= \frac{1}{n!}|\det(u_1 - u_0, \ldots, u_n - u_0)|. \end{aligned} \tag{7}$$

As an example of volume computation, let us evaluate the volume of a ball $B(0, r) = \{x \in \mathbb{R}^n : |x| \le r\}$. First note that by homogeniety, $\mathrm{vol}\left(B(0, r)\right) = \mathrm{vol}\left(B(0, 1)\right) := w_n$. The gamma function defined on $x > 0$ is given by $\Gamma(x) = \int_{t=0}^{\infty} t^{x-1}e^{-t}dt$. The beta function defined on $x > 0, y > 0$ is given by $B(x, y) = \int_{t=0}^{1} t^{x-1}(1 - t)^{y-1}dt$.

Claim 1.8.
(i) $B(x, y) = \frac{\Gamma(x)\Gamma(y)}{\Gamma(x+y)}$ *For all* $x, y > 0$
(ii) $\Gamma(\frac{1}{2}) = \sqrt{\pi}$.
(iii) $\Gamma(x + 1) = x\Gamma(x)$
(iv) $w_n = \frac{\pi^{\frac{n}{2}}}{\Gamma(\frac{n}{2}+1)}$.

Let ΔABC be a triangle in the plane with sides $a = |BC|, b = |AC|, c = |AB|$, and let $s = \frac{a+b+c}{2}$ be its semiperimeter. Heron's formula for the area of the triangle is the following

Theorem 1.9 (Heron, Archimedes)**.**

$$\mathrm{area}(\Delta ABC) = \sqrt{s(s - a)(s - b)(s - c)}. \tag{8}$$

Let $S = \mathrm{conv}\,\{u_0, \ldots, u_n\}$ be a n-simplex in $\mathbb{R}^n$. For $0 \le i, j \le n$, let $d_{ij} = |v_i - v_j|$. define a symmetric $(n + 2) \times (n + 2)$ matrix

$$D = \begin{bmatrix} 0 & d_{0,1}^2 & \cdots & d_{0,n-1}^2 & d_{0,n}^2 & 1 \\ d_{1,0}^2 & 0 & \cdots & d_{1,n-1}^2 & d_{1,n}^2 & 1 \\ \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\ d_{n-1,0}^2 & d_{n-1,1}^2 & \cdots & 0 & d_{n-1,n}^2 & 1 \\ d_{n,0}^2 & d_{n,1}^2 & \cdots & d_{n,n-1}^2 & 0 & 1 \\ 1 & 1 & \cdots & 1 & 1 & 0 \end{bmatrix}. \tag{9}$$

The following high dimensional extension of Heron's formula expresses the volume of S in terms of the pairwise distances of its vertices.

Theorem 1.10 (Cayley-Menger)**.**

$$\mathrm{vol}(S) = \left(\frac{(-1)^{n+1}}{2^n(n!)^2} \det D \right)^{\frac{1}{2}}. \tag{10}$$

Proof. Let

$$A = \begin{bmatrix} — & — & u_0^T & — & — & 1 & 0 \\ — & — & u_1^T & — & — & 1 & 0 \\ & & \vdots & & & \vdots & \vdots \\ — & — & u_n^T & — & — & 1 & 0 \\ 0 & 0 & \cdots & 0 & 0 & 0 & 1 \end{bmatrix}, \tag{11}$$

$$B = \begin{bmatrix} | & | & \cdots & | & | \\ | & | & \cdots & | & | \\ u_0 & u_1 & \cdots & u_n & 0 \\ | & | & \cdots & | & | \\ | & | & \cdots & | & | \\ 0 & 0 & \cdots & 0 & 1 \\ 1 & 1 & \cdots & 1 & 0 \end{bmatrix}. \tag{12}$$

Claim 1.11.

$$\begin{aligned} \det A &= (-1)^n \det(u_1 - u_0, \dots, u_n - u_0), \\ \det B &= (-1)^{n+1} \det(u_1 - u_0, \dots, u_n - u_0), \end{aligned}$$

and

$$AB = \begin{bmatrix} (u_0, u_0) & (u_0, u_1) & \cdots & (u_0, u_{n-1}) & (u_0, u_n) & 1 \\ (u_1, u_0) & (u_1, u_1) & \cdots & (u_1, u_{n-1}) & (u_1, u_n) & 1 \\ \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\ (u_{n-1}, u_0) & (u_{n-1}, u_1) & \cdots & (u_{n-1}, u_{n-1}) & (u_{n-1}, u_n) & 1 \\ (u_n, u_0) & (u_n, u_1) & \cdots & (u_n, u_{n-1}) & (u_n, u_n) & 1 \\ 1 & 1 & \cdots & 1 & 1 & 0 \end{bmatrix}. \tag{13}$$

Now,

$$(u_i, u_j) = \frac{1}{2}\left(|u_i|^2 + |u_j|^2 - d_{ij}^2\right).$$

Thus, by subtracting $\frac{|u_j|^2}{2}$ times of the last column from the j-th column, and then subtracting $\frac{|u_i|^2}{2}$ times of the last row from the i-th row, it follows that

$$\begin{aligned} \det(AB) &= \det \begin{bmatrix} 0 & -\frac{d_{0,1}^2}{2} & \cdots & -\frac{d_{0,n-1}^2}{2} & -\frac{d_{0,n}^2}{2} & 1 \\ -\frac{d_{1,0}^2}{2} & 0 & \cdots & -\frac{d_{1,n-1}^2}{2} & -\frac{d_{1,n}^2}{2} & 1 \\ \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\ -\frac{d_{n-1,0}^2}{2} & -\frac{d_{n-1,1}^2}{2} & \cdots & 0 & -\frac{d_{n-1,n}^2}{2} & 1 \\ -\frac{d_{n,0}^2}{2} & -\frac{d_{n,1}^2}{2} & \cdots & -\frac{d_{n,n-1}^2}{2} & 0 & 1 \\ 1 & 1 & \cdots & 1 & 1 & 0 \end{bmatrix} \\ &= 4 \cdot \left(-\frac{1}{2}\right)^{n+2} \det D = \frac{(-1)^n}{2^n} \det D. \end{aligned}$$

Therefore

$$\begin{aligned} (n!)^2 \mathrm{vol}(S)^2 &= \mathrm{vol}\left(P(u_1 - u_0, \dots, u_n - u_0)\right)^2 \\ &= \det(u_1 - u_0, \dots, u_n - u_0)^2 \\ &= -\det(AB) = \frac{(-1)^{n+1}}{2^n} \det D. \end{aligned}$$

□

The radius of the circumscribed circle of a triangle is given by

$$\frac{abc}{4\sqrt{s(s-a)(s-b)(s-c)}}.$$

The high dimensional extension is as follows. Let R be the radius of the sphere circumscrbing the simplex $S = \mathrm{conv}\,\{u_0, \ldots, u_n\}$ and let

$$D_0 = \begin{bmatrix} 0 & d_{0,1}^2 & \cdots & d_{0,n-1}^2 & d_{0,n}^2 \\ d_{1,0}^2 & 0 & \cdots & d_{1,n-1}^2 & d_{1,n}^2 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ d_{n-1,0}^2 & d_{n-1,1}^2 & \cdots & 0 & d_{n-1,n}^2 \\ d_{n,0}^2 & d_{n,1}^2 & \cdots & d_{n,n-1}^2 & 0 \end{bmatrix}. \tag{14}$$

Theorem 1.12. $R^2 = -\frac{\det(D_0)}{2\det(D)}$.

Proof. Let u_{n+1} denote the center of the circumscribing sphere of S. By adding an $(n+1)$ fixed coordinate, we may view $\{u_i\}_{i=0}^{n+1}$ as points in $\mathbb{R}^{n+1}$. Let

$$\tilde{D} = \begin{bmatrix} 0 & d_{0,1}^2 & \cdots & d_{0,n-1}^2 & d_{0,n}^2 & R^2 & 1 \\ d_{1,0}^2 & 0 & \cdots & d_{1,n-1}^2 & d_{1,n}^2 & R^2 & 1 \\ \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \vdots \\ d_{n-1,0}^2 & d_{n-1,1}^2 & \cdots & 0 & d_{n-1,n}^2 & R^2 & 1 \\ d_{n,0}^2 & d_{n,1}^2 & \cdots & d_{n,n-1}^2 & 0 & R^2 & 1 \\ R^2 & R^2 & \cdots & R^2 & R^2 & 0 & 1 \\ 1 & 1 & \cdots & 1 & 1 & 1 & 0 \end{bmatrix}. \tag{15}$$

The simplex $\tilde{S} = \mathrm{conv}\,\{u_i\}_{i=0}^{n+1}$ is degenerate and thus has volume 0. Noting that $d_{i,n+1}^2 = R^2$ for all $0 \le i \le n$, it follows from Theorem 1.10 that $\det \tilde{D} = 0$. Subtracting R^2 times the $(n+3)$-rd column from the $(n+2)$-nd column, and then subtracting R^2 times the $(n+3)$-rd row from the $(n+2)$-nd row, it follows that $0 = \det \tilde{D} = \det E$, where

$$E = \begin{bmatrix} 0 & d_{0,1}^2 & \cdots & d_{0,n-1}^2 & d_{0,n}^2 & 0 & 1 \\ d_{1,0}^2 & 0 & \cdots & d_{1,n-1}^2 & d_{1,n}^2 & 0 & 1 \\ \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \vdots \\ d_{n-1,0}^2 & d_{n-1,1}^2 & \cdots & 0 & d_{n-1,n}^2 & 0 & 1 \\ d_{n,0}^2 & d_{n,1}^2 & \cdots & d_{n,n-1}^2 & 0 & 0 & 1 \\ 0 & 0 & \cdots & 0 & 0 & -2R^2 & 1 \\ 1 & 1 & \cdots & 1 & 1 & 1 & 0 \end{bmatrix}. \tag{16}$$

Expanding $\det E$ by the $(n+2)$-nd row we obtain

$$0 = \det \tilde{D} = \det E = -\det D_0 - 2R^2 \det D,$$

hence $R^2 = -\frac{\det D_0}{2\det D}$.

□

1.3 Touching Spheres

Let $C_1 = S(a_1, r_1), \dots, C_{n+2} = S(a_{n+2}, r_{n+2})$ be pairwise touching (i.e. tangent) spheres in $\mathbb{R}^n$. The following result was proved by Descartes for $n = 2$ and by Gosset for general n.

Theorem 1.13 (Descartes, Gosset)**.**
(i) If all C_i's touch externally then

$$\left(\sum_{i=1}^{n+2} r_i^{-1}\right)^2 = n \sum_{i=1}^{n+2} r_i^{-2}. \tag{17}$$

(ii) If $C_1, \dots, C_{n+1}$ touch externally among themselves, and all touch C_{n+2} internally, then

$$\left(\sum_{i=1}^{n+1} r_i^{-1} - r_{n+2}^{-1}\right)^2 = n \sum_{i=1}^{n+2} r_i^{-2} \tag{18}$$

Proof. (i) The assumption that the C_i's are externally touching implies that for all $1 \le i, j \le n+2$

$$|a_i - a_j|^2 = (r_i + r_j)^2 - 4\delta_{ij} r_i r_j. \tag{19}$$

The $a_1, \dots, a_{n+2} \in \mathbb{R}^n$ are affinely dependent, i.e. there exists a $0 \ne (\beta_1, \dots, \beta_{n+2}) \in \mathbb{R}^{n+2}$ such that $\sum_{i=1}^{n+2} \beta_i = 0$ and $\sum_{i=1}^{n+2} \beta_i a_i = 0$. Multiplying (19) by β_i and over all i's we obtain

$$\sum_{i=1}^{n+2} \beta_i |a_i|^2 = \sum_{i=1}^{n+2} \beta_i r_i^2 + 2\left(\sum_{i=1}^{n+2} \beta_i r_i\right) r_j - 4\beta_j r_j^2. \tag{20}$$

Let

$$A = \sum_{i=1}^{n+2} \beta_i r_i \quad , \quad B = \sum_{i=1}^{n+2} \beta_i (r_i^2 - |a_i|^2).$$

Then (20) implies that for all $1 \le j \le n$

$$4 r_j^2 \beta_j = 2A r_j + B. \tag{21}$$

Dividing (21) by r_j and summing over all j's we obtain

$$4A = 2(n+2)A + B \sum_{j=1}^{n+2} \frac{1}{r_j}, \tag{22}$$

i.e.

$$A = -\frac{B}{2n} \sum_{j=1}^{n+2} \frac{1}{r_j}. \tag{23}$$

Dividing (21) by r_j^2 and summing over all j's we obtain

$$0 = 4 \sum_{j=1}^{n+2} \beta_j = 2A \sum_{j=1}^{n+2} \frac{1}{r_j} + B \sum_{j=1}^{n+2} \frac{1}{r_j^2}. \tag{24}$$

If $B = 0$ then $A = 0$ and hence $\beta_j = 0$ for all j's, a contradiction. Hence $B \ne 0$. Substituting (23) in (24) we obtain

$$0 = -\frac{B}{n} \left(\sum_{j=1}^{n+2} \frac{1}{r_j}\right)^2 + B \sum_{j=1}^{n+2} \frac{1}{r_j^2}. \tag{25}$$

Dividing by B we obtain (17).

$\square$

1.4 Inversion

Inversion is the map $\phi : \mathbb{R}^n \setminus \{0\} \to \mathbb{R}^n \setminus \{0\}$ given by $\phi(x) = \frac{x}{|x|^2}$. Clearly, ϕ is an involution, i.e. $\phi^2 = I$. In this section we will study some properties of this map. For $a \in \mathbb{R}^n$ and $r > 0$, let $S(a,r) = \{x \in \mathbb{R}^n : |x-a| = r\}$ be the sphere of radius r and center a. For $0 \neq u \in \mathbb{R}^n$ and $\alpha \in \mathbb{R}$ let $H_{u,\alpha} = \{x \in \mathbb{R}^n : (x,u) = \alpha\}$ be the hyperplane orthogonal to u that passes through the point $\frac{\alpha u}{|u|^2}$.

Claim 1.14.
(i) ϕ maps $H_{u,0} \setminus \{0\}$ bijectively onto itself.
(ii) If $\alpha \neq 0$ then ϕ map $H_{u,\alpha}$ bijectively onto $S\left(\frac{u}{2\alpha}, \frac{|u|}{2|\alpha|}\right) \setminus \{0\}$.
(iii) Let $S(a,r)$ be a sphere that contains 0. Then ϕ maps $S(a,r) \setminus \{0\}$ onto the hyperplane $H_{\frac{a}{2|a|^2}, \frac{1}{4|a|^2}}$.
(iv) Let $S(a,r)$ be a sphere with $|a| > r$. Then ϕ maps $S(a,r)$ bijectively onto the sphere $S\left(\frac{a}{|a|^2-r^2}, \frac{r}{|a|^2-r^2}\right)$.
(v) Let $S(a,r)$ be a sphere with $|a| < r$. Then ϕ maps $S(a,r)$ bijectively onto the sphere $S\left(\frac{a}{|a|^2-r^2}, \frac{r}{r^2-|a|^2}\right)$.

Proof. (i) is clear. (ii) Let $v \in H_{u,\alpha}$. Then

$$\begin{aligned} \left|\phi(v) - \frac{u}{2\alpha}\right|^2 &= \left|\frac{v}{|v|^2} - \frac{u}{2\alpha}\right|^2 \\ &= \frac{1}{|v|^2} + \frac{|u|^2}{4\alpha^2} - 2\left(\frac{v}{|v|^2}, \frac{u}{2\alpha}\right) \\ &= \frac{|u|^2}{4\alpha^2}. \end{aligned} \tag{26}$$

(iii) follows from (ii). To show (iv), suppose that $|a| > r$ and $x \in S(a,r)$. Then

$$\begin{aligned} \left|\phi(x) - \frac{a}{|a|^2 - r^2}\right|^2 &= \left|\frac{x}{|x|^2} - \frac{a}{|a|^2 - r^2}\right|^2 \\ &= \frac{1}{|x|^2} + \frac{|a|^2}{(|a|^2 - r^2)^2} - \frac{2(x,a)}{|x|^2\,(|a|^2 - r^2)} \\ &= \frac{|a|^2 - r^2 - 2(x,a)}{|x|^2\,(|a|^2 - r^2)} + \frac{|a|^2}{(|a|^2 - r^2)^2} \\ &= \frac{|a|^2}{(|a|^2 - r^2)^2} - \frac{|a|^2 - r^2}{(|a|^2 - r^2)^2} \\ &= \left(\frac{r}{|a|^2 - r^2}\right)^2. \end{aligned} \tag{27}$$

The proof of (v) is essentially the same.

□

2 Projective Geometry

2.1 The n-Dimensional Projective Space

Let V be an $(n+1)$-dimensional vector space over a field $\mathbb{F}$. The projective space $P(V)$ associated with V is defined as follows. The *points* of $P(V)$ are the 1-dimensional linear subspaces of V. Given a nonzero $u \in V$, let $[u] = \text{span}\{u\}$ denote the line spanned by u. A *projective subspace* (or flat) of $P(V)$ is the set $[U] = \{[u] : 0 \neq u \in U\}$, where U is a linear subspace of V. The *dimension* of $[U]$ is $\dim[U] = \dim U - 1$. When $V = \mathbb{F}^{n+1}$ and $0 \neq u = (a_0, \ldots, a_n) \in V$, we denote $[u]$ by its *homogenous coordinates* $[a_0, \ldots, a_n]$. The projective space $P(\mathbb{F}^{n+1})$ is denoted by $\mathbb{F}P^n$. Let V^* denote the dual space of V. This is the space of linear functionals of V. The *dual* of $P(V)$ is $P(V^*)$. The *duality map* Φ is the bijective map from the set of projective subspaces of $P(V)$ to projective subspaces of $P(V^*)$, that assigns to a subspace $P(U)$ of $P(V)$, the subspace $P(U^\circ)$, where

$$U^\circ = \{\phi \in V^* : \phi(u) = 0 \text{ for all } u \in U\}$$

is the *annihilator* of U. Note that $\Phi^2 = \text{Identity}$, and $\dim \Phi P(U) = \dim P(V) - \dim P(U)$. For example, if $dimV = 3$ then Φ maps points to lines, and lines to points. Furthermore, $P(U_1) \subset P(U_2)$ iff $\Phi P(U_1) \supset \Phi P(U_2)$. This leads to the following

Duality Principle: Any true statement concerning incidence in $P(V)$ gives rise to a true statement in $P(V^*)$ obtain by replacing any $P(U)$ by its dual $\Phi P(U)$. Consider, for example, the statement: any two distinct points in $\mathbb{F}P^2$ are contained in a unique line. The dual statement is: Any two distinct lines in $\mathbb{F}P^2$ intersect in a single point.

Let $GL(V)$ denote the *general linear group* of V, i.e. all invertible linear transformations of V. Let $Z(V) = \{cI : c \neq 0\}$ be the normal subgroup consisting of all nonzero multiples of the identity. The *projective linear group* is defined as $PGL(V) = GL(V)/Z(V)$. The action of $PGL(V)$ on $P(V)$ is given as follows. Let $g \in GL(V)$ and let $\overline{g}$ be its image in $PGL(V)$. Let $0 \neq u \in V$. Then $\overline{g}[v] = [gv]$. Clearly, if $[U]$ is a projective subspace of $P(V)$, then $T[U]$ is a projective subspace and $\dim T[U] = \dim[U]$ for any $T \in PGL(V)$. A set $A \subset P(V)$ is in *general position* such that C is linearly independent for any $C \subset A$ of cardinality $|C| \leq n+1$.

Claim 2.1. *Let $A = \{p_1, \ldots, p_{n+2}\}$ and $B = \{q_1, \ldots, q_{n+2}\}$ be two sets in general position in $P(V)$. Then there exists a unique projective transformation $T \in PGL(V)$ such that $Tp_i = q_i$ for all $1 \leq i \leq n+2$.*

Proof. For $1 \leq i \leq n+2$ write $p_i = [u_i]$ and $q_i = [v_i]$, where $u_i, v_i \in V$. Then both $u_1, \ldots, u_{n+1}$ and $v_1, \ldots, v_{n+2}$ are bases of V. There exist unique $(\alpha_1, \ldots, \alpha_{n+1}), (\beta_1, \ldots, \beta_{n+1} \in \mathbb{F}^{n+1}$ such that $u_{n+2} = \sum_{i=1}^{n+1} \alpha_i u_i$ and $v_{n+2} = \sum_{i=1}^{n+1} \beta_i v_i$. By general position, $\alpha_i \neq 0$ and $\beta_i \neq 0$ for all $1 \leq i \leq n+1$. Let $g \in GL(V)$ be given by $g(\alpha_i u_i) = \beta_i v_i$. Then $T = \overline{g}$ satisfies $Tp_i = q_i$ for all $1 \leq i \leq n+1$. Furthermore

$$\begin{aligned} Tp_{n+2} = [gu_{n+2}] &= \left[g \sum_{i=1}^{n+1} \alpha_i u_i\right] = \left[\sum_{i=1}^{n+1} g(\alpha_i u_i)\right] \\ &= \left[\sum_{i=1}^{n+1} \beta_i v_i\right] = [v_{n+2}] = q_{n+2}. \end{aligned} \tag{28}$$

For the uniqueness, assume that $Sp_i = q_i$ for $1 \leq i \leq n+2$, for some $S = \overline{h}$. Then there exists nonzero $\gamma_1, \ldots, \gamma_{n+2}$ such that $h(\alpha_i u_i) = \gamma_i v_i$. Then

$$\begin{aligned} \left[\sum_{i=1}^{n+1} \beta_i v_i\right] &= q_{n+2} = Sp_{n+2} = [hu_{n+2}] \\ &= \left[h\left(\sum_{i=1}^{n+1} \alpha_i u_i\right)\right] = \left[\sum_{i=1}^{n+1} h(\alpha_i u_i)\right] = \left[\sum_{i=1} \gamma_i v_i\right]. \end{aligned} \tag{29}$$

It follows that there exists a $\theta \neq 0$ such that $\gamma_i = \theta \beta_i$ for all $1 \leq i \leq n+1$. Therefore

$$h(\alpha_i u_i) = \gamma_i u_i = \theta \beta_i v_i = \theta g(\alpha_i u_i).$$

Thus $S = \overline{h} = \overline{g} = T$.

□

2.2 Desargue and Pappus Theorems

Theorem 2.2 (Desargue)**.** *Let L_1, L_2, L_3 be three distinct lines in $\mathbb{F}P^2$ that intersect in a point p. Consider three pairs of distinct points $a_1, a_2 \in L_1 \setminus \{p\}$, $b_1, b_2 \in L_2 \setminus \{p\}$ and $c_1, c_2 \in L_3 \setminus \{p\}$. Let $q_1 = \overline{b_1c_1} \cap \overline{b_2c_2}$, $q_2 = \overline{a_1c_1} \cap \overline{a_2c_2}$ and $q_3 = \overline{a_1b_1} \cap \overline{a_2b_2}$. Then q_1, q_2, q_3 are collinear.*

Proof. We identify a projective point with any of its representatives in $\mathbb{F}^3 \setminus \{0\}$. There exist $\alpha_1, \alpha_2, \beta_1, \beta_2, \gamma_1, \gamma_2 \in \mathbb{F}$ such that

$$p = \alpha_1 a_1 + \alpha_2 a_2 = \beta_1 b_1 + \beta_2 b_2 = \gamma_1 c_1 + \gamma_2 c_2.$$

It follows that $\alpha_1 a_1 - \beta_1 b_1 = \beta_2 b_2 - \alpha_2 a_2$. As $\alpha_1 a_1 - \beta_1 b_1 \in \overline{a_1 b_1}$ and $\beta_2 b_2 - \alpha_2 a_2 \in \overline{a_2 b_2}$, it follows that $q_3 = \alpha_1 a_1 - \beta_1 b_1$. Similarly $q_1 = \beta_1 b_1 - \gamma_1 c_1$ and $q_2 = \gamma_1 c_1 - \alpha_1 a_1$. Summing the three equalities, we obtain that $q_1 + q_2 + q_3 = 0$. Hence q_1, q_2, q_3 are collinear.

□

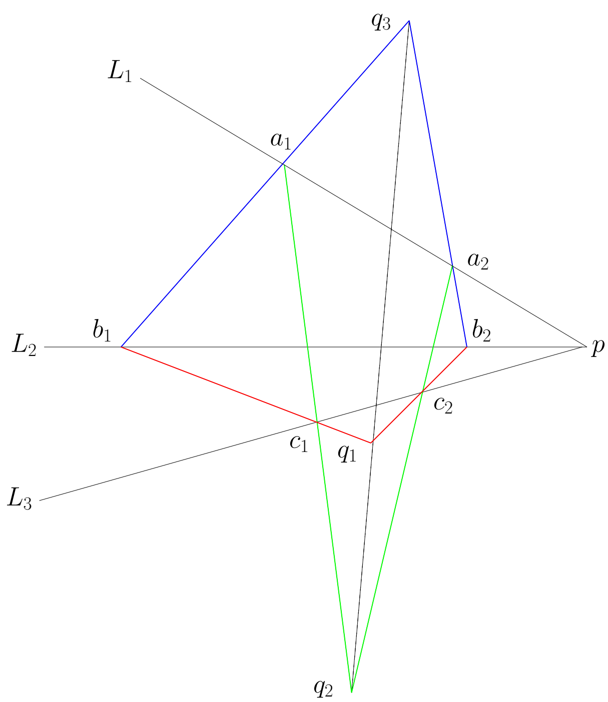

Figure 5: Desargue Theorem

Remark: It turns out that the dual to Desargue theorem is the converse to Desargue theorem (check!).

Theorem 2.3 (Pappus)**.** *Let L_1, L_2 be two distinct lines in $\mathbb{FP}^2$ and let $q = L_1 \cap L_2$. Let a_1, a_2, a_3 be distinct points in $L_1 \setminus \{q\}$, and let b_1, b_2, b_3 be distinct points in $L_2 \setminus \{q\}$. Let $c_1 = \overline{a_2b_3} \cap \overline{a_3b_2}$, $c_2 = \overline{a_1b_3} \cap \overline{a_3b_1}$ and $c_3 = \overline{a_1b_2} \cap \overline{a_2b_1}$. Then c_1, c_2, c_3 are collinear.*

Proof. Without loss of generality, the points a_1, b_1, c_1, a_3 are in general position. We may therefore assume that $a_1 = [1, 0, 0]$, $[b_1 = [0, 1, 0]$, $c_1 = 0, 0, 1]$ and $a_3 = [1, 1, 1]$. It follows that

$$\begin{aligned} \Phi(\overline{a_2c_1}) &= [1, -p, 0], \\ \Phi(\overline{a_1c_2}) &= [0, 1, -q], \\ \Phi(\overline{b_1b_2}) &= [-r, 0, 1]. \end{aligned} \tag{30}$$

As

$$\overline{a_2c_1} \cap \overline{a_1c_2} \cap \overline{b_1b_2} = b_3,$$

it follows that

$$0 = \det \begin{bmatrix} 1 & -p & 0 \\ 0 & 1 & -q \\ -r & 0 & 1 \end{bmatrix} = 1 - pqr.$$

Therefore $pqr = 1$. Now

$$\begin{aligned} \Phi(\overline{a_2 b_1}) &= [1, 0, -p], \\ \Phi(\overline{a_1 b_2}) &= [0, -r, 1], \\ \Phi(\overline{c_1 c_2}) &= [-q, 1, 0]. \end{aligned} \tag{31}$$

Now

$$\det \begin{bmatrix} 1 & 0 & -p \\ 0 & -r & 1 \\ -q & 1 & 0 \end{bmatrix} = -1 + pqr = 0.$$

It follows that

$$\{c_3\} \cap \overline{c_1 c_2} = \left(\overline{a_2 b_1} \cap \overline{a_1 b_2}\right) \cap \overline{c_1 c_2}).$$

Thus c_1, c_2, c_3 are collinear.

□

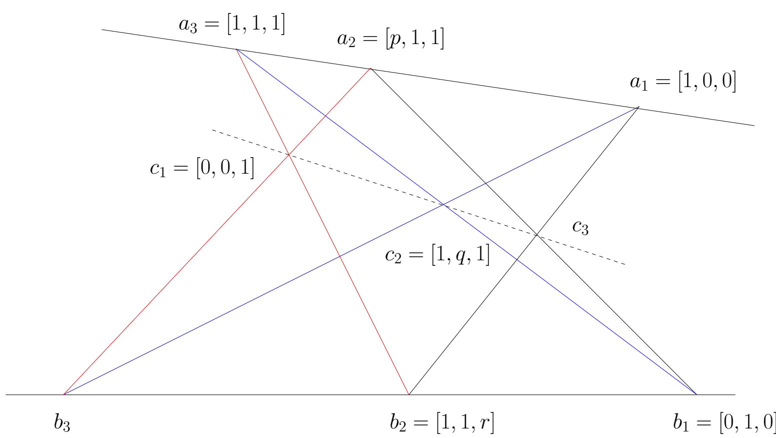

Figure 6: Pappus Theorem: first proof

We next give a different proof of Pappus theorem. We need some preliminaries. Let $A \in M_n(\mathbb{F})$. For subsets $I = \{i_1 < \cdots < i_k\}, J = \{j_1 < \cdots < j_k\} \subset [n]$, let $B = A[I, J] \in M_k(\mathbb{F})$ be given by $B_{st} = A_{i_s j_t}$. For a subset $K \subset [n]$, let $\overline{K} = [n] \setminus K$. For a partition $[n] = I \cup \overline{I}$ where $I = \{i_1 < \cdots < i_k\}$, $\overline{I} = \{j_1 < \cdots < j_{n-k}\}$, let

$$\sigma_{I,\overline{I}} = \begin{pmatrix} 1 & \cdots & k & k+1 & \cdots & n \\ i_1 & \cdots & i_k & j_1 & \cdots & j_{n-k} \end{pmatrix}.$$

Proposition 2.4 (Laplace Expansion)**.** *Let $K \cup \overline{K}$ be a partition of $[n]$ with $|K| = k$, and let $A \in M_n(\mathbb{F})$. Then*

$$\det A = \sum_{I \in \binom{[n]}{k}} sgn\left(\sigma_{I,\overline{I}}\right) \det A[K, I] \cdot \det A[\overline{K}, \overline{I}]. \tag{32}$$

For vectors $u_1, \ldots, u_n \in \mathbb{F}^n$ we abbreviate $[u_1, \ldots, u_n] = \det(u_1, \ldots, u_n)$.

Proposition 2.5 (Plücker relation)**.** *Let $u_1, \ldots, u_4 \in \mathbb{F}^2$. Then*

$$[u_1, u_2] \cdot [u_3, u_4] - [u_1, u_3] \cdot [u_2, u_4] + [u_1, u_4] \cdot [u_2, u_3] = 0. \tag{33}$$

Proof. Let

$$A = \begin{bmatrix} u_1 & u_2 & u_3 & u_4 \\ u_1 & u_2 & u_3 & u_4 \end{bmatrix} \in M_4(\mathbb{F}).$$

Then $\mathrm{rk}(A) \leq 2$, so in particular $\det A = 0$. On the other hand, by Proposition 2.4:

$$\det A = 2\left([u_1, u_2] \cdot [u_3, u_4] - [u_1, u_3] \cdot [u_2, u_4] + [u_1, u_4] \cdot [u_2, u_3]\right).$$

□

Corollary 2.6. *Let $u_0, \ldots, u_4 \in \mathbb{F}^3$. Then:*

$$[u_0, u_1, u_2] \cdot [u_0, u_3, u_4] - [u_0, u_1, u_3] \cdot [u_0, u_2, u_4] + [u_0, u_1, u_4] \cdot [u_0, u_2, u_3] = 0. \quad (34)$$

Proof. For $u_0 = e_1$, Eq. (34) follows from (33). In general, Let $T \in GL_3(\mathbb{F})$ such that $Tu_0 = e_1$. Then

$$\begin{aligned} 0 &= [Tu_0, Tu_1, Tu_2] \cdot [Tu_0, Tu_3, Tu_4] - [Tu_0, Tu_1, Tu_3] \cdot [Tu_0, Tu_2, Tu_4] + [Tu_0, Tu_1, Tu_4] \cdot [Tu_0, Tu_2, Tu_3] \\ &= \det(T)^2 \left([u_0, u_1, u_2] \cdot [u_0, u_3, u_4] - [u_0, u_1, u_3] \cdot [u_0, u_2, u_4] + [u_0, u_1, u_4] \cdot [u_0, u_2, u_3]\right). \end{aligned}$$

□

Another proof of Pappus Theorem.

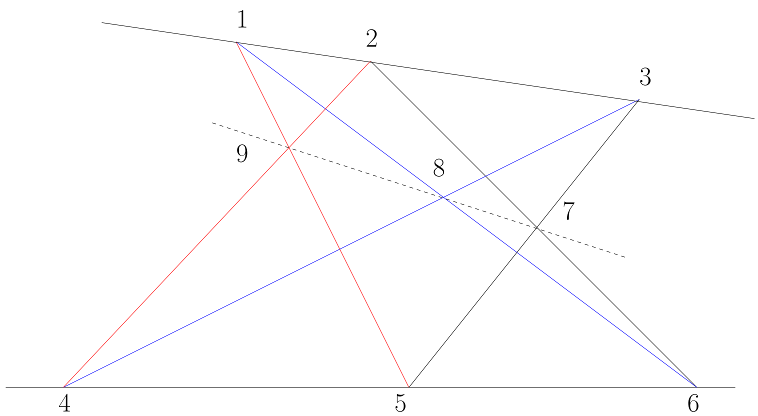

Figure 7: Pappus Theorem: second proof

Using the Plücker relation for each of the 8 lines in the configuration we obtain:

$$\begin{aligned}
&[147] \cdot [123] - [142] \cdot [173] + [143] \cdot [172] = 0 \Rightarrow [142] \cdot [173] = [143] \cdot [172], \\
&[147] \cdot [159] - [145] \cdot [179] + [149] \cdot [175] = 0 \Rightarrow [145] \cdot [179] = [149] \cdot [175], \\
&[147] \cdot [186] - [148] \cdot [176] + [146] \cdot [178] = 0 \Rightarrow [148] \cdot [176] = [146] \cdot [178], \\
&[471] \cdot [456] - [475] \cdot [416] + [476] \cdot [415] = 0 \Rightarrow [475] \cdot [416] = [476] \cdot [415], \\
&[471] \cdot [483] - [478] \cdot [413] + [473] \cdot [418] = 0 \Rightarrow [478] \cdot [413] = [473] \cdot [418], \\
&[471] \cdot [429] - [472] \cdot [419] + [479] \cdot [412] = 0 \Rightarrow [472] \cdot [419] = [479] \cdot [412], \\
&[714] \cdot [726] - [712] \cdot [746] + [716] \cdot [742] = 0 \Rightarrow [712] \cdot [746] = [716] \cdot [742], \\
&[714] \cdot [753] - [715] \cdot [743] + [713] \cdot [745] = 0 \Rightarrow [715] \cdot [743] = [713] \cdot [745].
\end{aligned}$$

Multiplying the columns and cancelling equal terms, we obtain

$$[719] \cdot [748] = [718] \cdot [749].$$

On the other hand

$$[714] \cdot [789] - [719] \cdot [748] + [718] \cdot [749] = 0.$$

Hence $[789] = 0$ and therefore $7, 8, 9$ are collinear.

□

2.3 Conic Sections and Quadratic Forms

Consider the cone $C = \{(x, y, z) \in \mathbb{R}^3 : z^2 = x^2 + y^2\}$. A *conic section* is the intersection of C with a plane H. There are several possibilities.

1. $H = e_3^{\perp}$. Then $H \cap C$ is the single point $\{0\}$.
2. $H = e_2^{\perp}$. Then $H \cap C$ is the two intersecting lines $\operatorname{span}(e_1 - e_3) \cup \operatorname{span}(e_1 + e_3)$.
3. $H = (e_1 - e_3)^{\perp}$. Then $H \cap C$ is the line $\operatorname{span}\{e_1 + e_3)$.
4. $H = H_{e_1,1}$. Then $H \cap C$ is the hyperbola $\{(1, y, z) : 1 + y^2 = z^2\}$.
5. $H = H_{e_1+e_3,-1}$. Then $H \cap C$ is the parabola $\{\frac{y^2-1}{2}, y, -\frac{y^2+1}{2}) : y \in \mathbb{R}\}$.
6. $H = H_{e_3,1}$. Then $H \cap C$ is the circle $\{(x, y, 1) : x^2 + y^2 = 1\}$.

Let A be a real symmetric matrix in $M_n(\mathbb{R})$. The associated quadratic form is $q_A(x) = x^T Ax$. Two symmetric matrices A and B are *congruent* if there exists a matrix $P \in GL_n(\mathbb{R})$ such that $B = PAP^T$. Let n_+ (n_-) be the number of positive (negative) eigenvalues of A. The *signature* of A is (n_+, n_-).

Proposition 2.7 (Sylvester's Inertia Theorem)**.** *Let $A, B \in M_n(\mathbb{R})$ be symmetric. Then:*
(i) A is congruent to a diagonal matrix $D_{n_+,n_-} = \operatorname{diag}(d_1, \dots, d_n)$ where

$$d_i = \begin{cases} +1 & 1 \le i \le n_+ \\ -1 & n_+ + 1 \le i \le n_+ + n_- \\ 0 & n_+ + n_- + 1 \le i \le n. \end{cases}$$

(ii) A and B are congruent iff they have the same rank and the same signature.

Proof. (i) Let n_+, n_- denote respectively the number of positive and negative eigenvalues of A. Let $\{\lambda_i\}_{i=1}^n$ denote the eigenvalues of A. We may assume that $\lambda_i > 0$ for $1 \le i \le n_+$ and $\lambda_i < 0$ for $n_+ + 1 \le i \le n_+ + n_-$. Let $S \in O(n)$ such that $SAS^{-1} = \operatorname{diag}(\lambda_1, \dots, \lambda_n)$. Let

$$\mu_i = \begin{cases} \frac{1}{\sqrt{\lambda_i}} & 1 \le i \le n_+ \\ \frac{1}{\sqrt{-\lambda_i}} & n_+ + 1 \le i \le n_+ + n_- \\ 1 & n_+ + n_- + 1 \le i \le n. \end{cases}$$

and let $T = \operatorname{diag}(\mu_1, \dots, \mu_n)S$. Then TAT^t is of the required form. (ii) It suffices to show that if $A = D_{n_+,n_-}$ is congruent to $B = D_{m_+,m_-}$ then $(n_+, n_-) = (m_+, m_-)$. Clearly, $n_+ + n_- = \operatorname{rk}(A) = \operatorname{rk}(B) = m_+ + m_-$. Write $B = TAT^t$, and let $u_1, \dots, u_n$ be the rows of T. Then

$$\sum_{i=1}^{m_+} x_i^2 - \sum_{i=m_++1}^{m_++m_-} x_i^2 = \sum_{j=1}^{n_+} (u_j \cdot x)^2 - \sum_{j=n_++1}^{n_++n_-} (u_j \cdot x)^2. \tag{35}$$

Suppose for contradiction that $(n_+, n_-) \neq (m_+, m_-)$. We may assume that $n_+ > m_+$. As $m_+ + (n - n_+) < n$, there exists a nonzero $\theta = (\theta_1, \dots, \theta_n) \in \mathbb{R}^n$ such that $\theta_i = 0$ for $1 \le i \le m_+$ and $u_j \cdot \theta = 0$ for $j > n_+$. It follows that the left hand side of (35) is negative, while the right hand side of (35) is non-negative, a contradiction.

□

Let $A \in M_{n+1}(\mathbb{R})$ be a symmetric matrix with a quadratic form $q_A(x)$. The corresponding quadratic $(n-1)$-dimensional surface is $Q_A = \{[x] \in \mathbb{RP}^n : q_A(x) = 0\}$. Sets $S_1, S_2 \subset \mathbb{P}(V)$ are *projectively equivalent* if there exists a projective transformation $T \in PGL(V)$ such that $TS_1 = S_2$.

Claim 2.8. *Q_A and Q_B are projectively equivalent iff A is congruent to either B or to $-B$.*

Proof. If $B = P^T AP$ for $P \in GL_{n+1}(\mathbb{R})$ then

$$[x] \in Q_B \Leftrightarrow x^T Bx = 0 \Leftrightarrow (Px)^T A(Px) = 0 \Leftrightarrow Px \in Q_A \Leftrightarrow [x] \in P^{-1}Q_A.$$

Hence $Q_B = P^{-1}Q_A$. If $-B = P^T AP$ then $Q_B = Q_{-B} = P^{-1}Q_A$. Conversely, suppose $Q_B = TQ_A$ for some $T \in PGL_{n+1}(\mathbb{R})$. It follows that there exist two bases $u_1, \dots, u_{n+1}$ and $v_1, \dots, v_{n+1}$ of $\mathbb{R}^{n+1}$ such that

$$\sum_{i=1}^{k}(u_i x)^2 - \sum_{i=k+1}^{k+\ell}(u_i x)^2 = 0 \Leftrightarrow \sum_{i=1}^{k'}(v_i x)^2 - \sum_{i=k'+1}^{k'+\ell'}(v_i x)^2 = 0. \tag{36}$$

By replacing B with $-B$ if necessary, we may assume that $k \geq \ell$ and $k' \geq \ell'$. We claim that $k = k'$ and $\ell = \ell'$ and therefore A is congruent to B. Indeed, suppose to the contrary that $k > k'$. Then there exists a nonzero $x \in \mathbb{R}^{n+1}$ such that

- $v_i x = v_{i+k'} x$ for $1 \leq i \leq \ell'$.
- $v_i x = 0$ for $\ell' + 1 \leq i \leq k'$.
- $u_i x = 0$ for $k + 1 \leq i \leq n + 1$.

It follows that there exists an $1 \leq i_0 \leq k$ such that $u_i x \neq 0$, and then the right hand side of (36) is zero, while the left hand side is positive, a contradiction. Thus $k = k'$. The proof that $\ell = \ell'$ is similar.

□

Using Proposition 2.7 and Claim 2.8, quadrics in $\mathbb{P}^2\mathbb{R}$ are classified as follows:

n_+	n_-	equation	quadric
1	0	$x^2 = 0$	double line: $x = 0$
1	1	$x^2 - y^2 = 0$	two lines: $x = y$ and $x = -y$
3	0	$x^2 + y^2 + z^2 = 0$	$\emptyset$
2	1	$x^2 + y^2 = z^2$	ellipse

Proposition 2.9. *Let $S = \{p_1, \ldots, p_5\}$ be 5 points in general position in $\mathbb{P}^2(\mathbb{R})$. Then there exists a unique quadric Q_A that contains S.*

Proof. By Claim 2.1 we may assume that $p_i = [e_i]$ for $i = 1, 2, 3$ and $p_4 = [1, 1, 1]$. Let $p_5 = [\alpha, \beta, \gamma]$. Then $\alpha\beta\gamma \neq 0$ and the unique quadric containing S is

$$f(x, y, z) = (\beta - \gamma)\alpha yz + (\gamma - \alpha)\beta xz + (\alpha - \beta)\gamma xy.$$

□

2.4 Cross-Ratios and Pascal's Theorem

Let U be 2-dimensional space over $\mathbb{F}$, and let $p, q, r, s \in P(U)$ such that p, q, r are distinct. The *cross-ratio* $(p, q; r, s)$ is defined as follows. Write $p = [u], q = [v], r = [w], s = [w']$. Let $w = \alpha u + \beta v$ and $w' = \alpha' u + \beta' v$. Note that $\alpha, \beta \neq 0$, and if $\beta' = 0$ then $\alpha' \neq 0$. Then

$$(p, q; r, s) = \frac{\alpha' \cdot \beta}{\alpha \cdot \beta'} \in \mathbb{F} \cup \{\infty\}. \tag{37}$$

It is straightforward to check that $(p, q; r, s)$ is well defined and that if $T \in PGL(V)$ then $(Tp, Tq; Tr, Ts) = (p, q; r, s)$. Pick an arbitrary basis z_1, z_2 for U. For an element $u = \lambda_1 z_1 + \lambda_2 z_2$, let $\varphi(u) = \frac{\lambda_1}{\lambda_2} \in \mathbb{F} \cup \{\infty\}$.

Claim 2.10.

$$(p, q; r, s) = \frac{(\varphi(p) - \varphi(r)) \cdot (\varphi(q) - \varphi(s))}{(\varphi(p) - \varphi(s))(\varphi(q) - \varphi(r))}. \tag{38}$$

Proof. Let $p = [u], q = [v], r = [w], s = [w']$. Write $u = a_1 z_1 + a_2 z_2$, $v = b_1 z_1 + b_2 z_2$, $w = c_1 z_1 + c_2 z_2$, and $w' = d_1 z_1 + d_2 z_2$. Let $r = \alpha u + \beta v$ and $s = \alpha' u + \beta' v$. Then

$$\begin{aligned} c_1 z_1 + c_2 z_2 = w = \alpha u + \beta v &= \alpha(a_1 z_1 + a_2 z_2) + \beta(b_1 z_1 + b_2 z_2) \\ &= (a_1 \alpha + b_1 \beta) z_1 + (a_2 \alpha + b_2 \beta) z_2. \end{aligned} \tag{39}$$

and

$$\begin{aligned} d_1 z_1 + d_2 z_2 = w' = \alpha' u + \beta' v &= \alpha'(a_1 z_1 + a_2 z_2) + \beta'(b_1 z_1 + b_2 z_2) \\ &= (a_1 \alpha' + b_1 \beta') z_1 + (a_2 \alpha' + b_2 \beta') z_2. \end{aligned} \tag{40}$$

It follows that

$$\begin{bmatrix} c_1 \\ c_2 \end{bmatrix} = \begin{bmatrix} a_1 & b_1 \\ a_2 & b_2 \end{bmatrix} \cdot \begin{bmatrix} \alpha \\ \beta \end{bmatrix} \quad , \quad \begin{bmatrix} d_1 \\ d_2 \end{bmatrix} = \begin{bmatrix} a_1 & b_1 \\ a_2 & b_2 \end{bmatrix} \cdot \begin{bmatrix} \alpha' \\ \beta' \end{bmatrix}. \tag{41}$$

Using Cramer's rule it follows that

$$\begin{aligned} \frac{\alpha}{\beta} &= \frac{\det \begin{bmatrix} c_1 & b_1 \\ c_2 & b_2 \end{bmatrix}}{\det \begin{bmatrix} a_1 & c_1 \\ a_2 & c_2 \end{bmatrix}} = \frac{c_1 b_2 - c_2 b_1}{a_1 c_2 - a_2 c_1}, \\ \frac{\alpha'}{\beta'} &= \frac{\det \begin{bmatrix} d_1 & b_1 \\ d_2 & b_2 \end{bmatrix}}{\det \begin{bmatrix} a_1 & d_1 \\ a_2 & d_2 \end{bmatrix}} = \frac{d_1 b_2 - d_2 b_1}{a_1 d_2 - a_2 d_1}. \end{aligned} \tag{42}$$

Hence

$$
\begin{aligned}
\frac{(\varphi(p)-\varphi(r))\cdot(\varphi(q)-\varphi(s))}{(\varphi(p)-\varphi(s))(\varphi(q)-\varphi(r))} &= \frac{\left(\frac{a_1}{a_2}-\frac{c_1}{c_2}\right)\cdot\left(\frac{b_1}{b_2}-\frac{d_1}{d_2}\right)}{\left(\frac{a_1}{a_2}-\frac{d_1}{d_2}\right)\cdot\left(\frac{b_1}{b_2}-\frac{c_1}{c_2}\right)} \\
&= \frac{(a_1c_2-a_2c_1)\cdot(b_1d_2-b_2d_1)}{(a_1d_2-a_2d_1)\cdot(b_1c_2-b_2c_1)} \\
&= \frac{(a_1c_2-a_2c_1)\cdot(b_1d_2-b_2d_1)}{(a_1d_2-a_2d_1)\cdot(b_1c_2-b_2c_1)} \\
&= \frac{\alpha'\beta}{\alpha\beta'} = (p,q;r,s).
\end{aligned} \tag{43}
$$

□

Proposition 2.11. *Let a,b,c,d distinct points on a line ℓ, and let $p \notin \ell$, all in the projective plane $\mathbb{P}^2(\mathbb{F})$. Let $\overline{pa},\overline{pb},\overline{pc},\overline{pd}$ be the four lines through p, and let a_1,b_1,c_1,d_1 be their dual points. Then $(a_1,b_1;c_1,d_1)=(a,b;c,d)$. In particular, if $\tilde{\ell}$ is another line that does not contain p, and $\ell'\cap\overline{pa}=a'$, $\ell'\cap\overline{pb}=b'$, $\ell'\cap\overline{pc}=c'$, $\ell'\cap\overline{pd}=d'$, then $(a,b;c,d)=(a',b';c',d')$.*

Proof. Write $p=[w]$, $a=[u]$, $b=[v]$, $c=[\alpha u+\beta v]$, $d=[\alpha' u+\beta' v]$. Let $a_1=[u_1]$, $b_1=[v_1]$. Thus $u_1\perp u,w$ and $v_1\perp v,w$. It follows that $c_1=\alpha_1u_1+\beta_1v_1$ and $d_1=\alpha_1'u_1+\beta_1'v_1$, where $\alpha_1=\alpha(v_1\cdot u)$, $\beta_1=-\beta(u_1\cdot v)$, and $\alpha_1'=\alpha'(v_1\cdot u)$, $\beta_1'=-\beta'(u_1\cdot v)$ (check!). It follows that

$$(a_1,b_1;c_1,d_1)=\frac{\alpha_1'\beta_1}{\alpha_1\beta_1'}=\frac{\alpha'\beta}{\alpha\beta'}=(a,b;c,d).$$

□

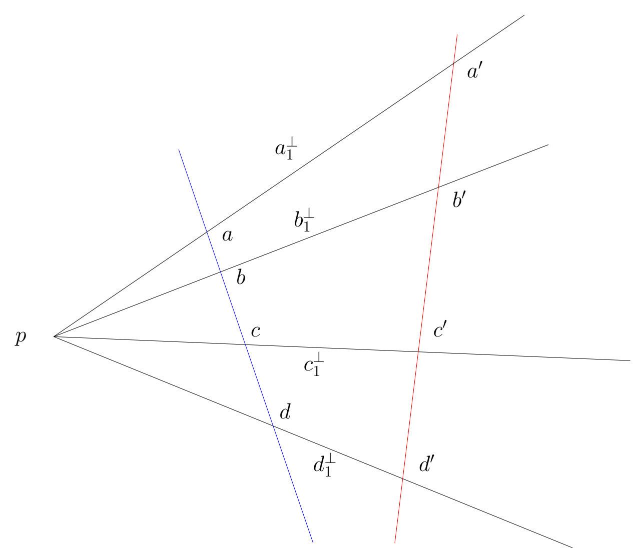

Figure 8: $(a,b;c,d)=(a_1,b_1;c_1,d_1)=(a',b';c',d')$

For four intersecting lines $\ell_1,\ell_2,\ell_3,\ell_4$ in $\mathbb{P}^2(\mathbb{F})$, define

$$(\ell_1,\ell_2;\ell_3,\ell_4)=(\Phi(\ell_1),\Phi(\ell_2);\Phi(\ell_3),\Phi(\ell_4)).$$

The dual form of Proposition 2.11 is the following

Proposition 2.12. *Let a,b,c,d be points on a line ℓ and let $p,q\notin\ell$. Then*

$$(\overline{pa},\overline{pb};\overline{pc},\overline{pd})=(a,b;c,d)=(\overline{qa},\overline{qb};\overline{qc},\overline{qd}).$$

The following result is a version of Proposition 2.12, where the degenerate conic ℓ is replaced by a nonsingular conic.

Proposition 2.13. *Let p, q, a, b, c, d be distinct points on a nonsingular conic C. Then*

$$(\overline{pa}, \overline{pb}; \overline{pc}, \overline{pd}) = (\overline{qa}, \overline{qb}; \overline{qc}, \overline{qd}). \tag{44}$$

Proof. Let $\overline{pa} = u_a^\perp$, $\overline{pb} = u_b^\perp$, $\overline{pc} = (u_a + u_b)^\perp$, $\overline{qa} = v_a^\perp$, $\overline{qb} = v_b^\perp$, $\overline{qc} = (v_a + v_b)^\perp$. It follows that $C = \{[x] \in \mathbb{P}^2(\mathbb{R}) : f(x) = 0\}$ where

$$f(x) = (u_a x) \cdot (v_b x) - (u_b x) \cdot (v_a x).$$

Let $\overline{pd} = (\alpha u_a + \beta u_b)^\perp$ and $\overline{qd} = (\alpha' v_a + \beta' v_b)^\perp$. Then $\alpha(u_a d) + \beta(u_b d) = 0$ and $\alpha'(v_a d) + \beta'(v_b d) = 0$. On the other hand $(u_a d) \cdot (v_b d) = (u_b d)(v_b d)$. Therefore

$$(\overline{pa}, \overline{pb}; \overline{pc}, \overline{pd}) = \frac{\alpha}{\beta} = \frac{\alpha'}{\beta'} = (\overline{qa}, \overline{qb}; \overline{qc}, \overline{qd}).$$

□

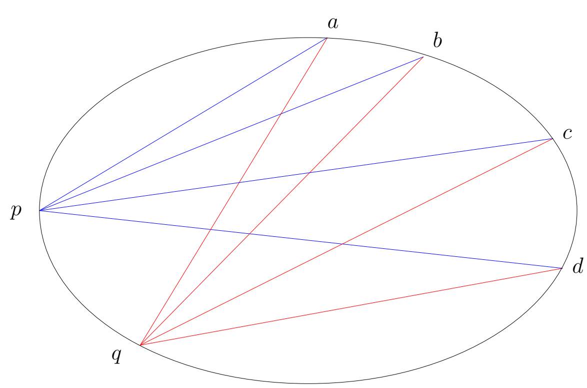

Figure 9: $(\overline{pa}, \overline{pb}; \overline{pc}, \overline{pd}) = (\overline{qa}, \overline{qb}; \overline{qc}, \overline{qd})$

Theorem 2.14 (Pascal)**.** *Let C be a nonsingluar quadric in $\mathbb{P}^2(\mathbb{R})$, and let $a_1, a_2, a_3, b_1, b_2, b_3$ be distinct points on C. Then the points $c_1 = a_3b_2 \cap a_2b_3$, $c_2 = a_1b_3 \cap a_3b_1$ and $c_3 = a_1b_2 \cap a_3b_2$ are collinear.*

Proof. Let $s = a_3b_2 \cap a_1b_3$, $t = a_3b_1 \cap a_1b_2$. Then

$$\begin{aligned}(a_3, c_1; s, b_2) &= (b_3a_3, b_3c_1; b_3s, b_3b_2) = (b_3a_3, b_3a_2; b_3a_1, b_3b_2) \\ &= (b_1a_3, b_1a_2; b_1a_1, b_1b_2) = (b_1t, b_1c_3; b_1a_1, b_1b_2) \\ &= (t, c_3; a_1, b_2).\end{aligned} \tag{45}$$

Therefore

$$(c_2a_3, c_2c_1; c_2s, c_2b_2) = (c_2t, c_2c_3; c_2a_1, c_2b_2) = (c_2a_3, c_2c_3; c_2s, c_2b_2). \tag{46}$$

It follows that $c_2c_1 = c_2c_3$, i.e. c_1, c_2, c_3 are collinear.

□

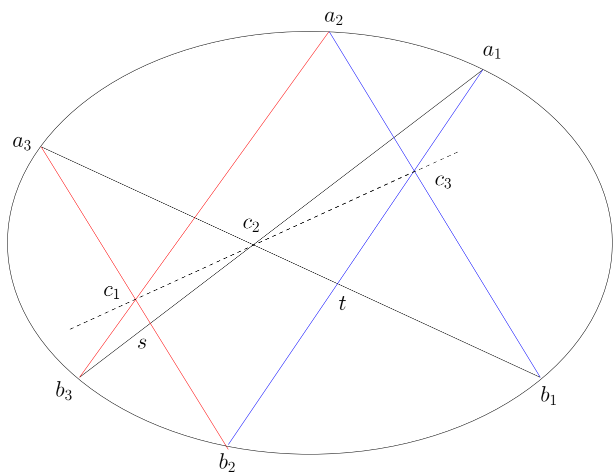

Figure 10: Pascal Theorem: c_1, c_2, c_3 are collinear

2.5 Metric Aspects of $\mathbb{P}^2(\mathbb{R})$

Let $L(\gamma)$ denote the length of a path $\gamma : [0,1] \to \mathbb{R}^3$. Define a metric on the unit sphere S^2 by $d(u,v) = \min L(\gamma)$ where γ ranges over all $\gamma : [0,1] \to S^2$ such that $\gamma(0) = u, \gamma(1) = v$.

Claim 2.15. *$d(u,v) = \arccos(u \cdot v)$.*

Proof. We may assume that $u = (0,0,1)$ and $v = (\sin\alpha, 0, \cos\alpha)$. Let γ be a path in S^2 between u and v, and write

$$\gamma(t) = (\sin(\phi(t))\cos(\theta(t)), \sin(\phi(t))\sin(\theta(t)), \cos(\phi(t))),$$

where $\phi(0) = 0, \phi(1) = \alpha$, $\theta(0) = \theta(1) = 0$. Then

$$\begin{aligned}L(\gamma) &= \int_{t=0}^{1} |\dot\gamma(t)| dt = \int_{t=0}^{1} \sqrt{\dot\phi(t)^2 + \theta(t)^2 \sin^2\phi(t)} dt \\ &\geq \int_{t=0}^{1} |\dot\phi(t)| dt \geq \int_{t=0}^{1} \dot\phi(t) dt = \phi(1) - \phi(0) = \alpha.\end{aligned} \tag{47}$$

□

For a point $u \in S^2$, let $A_u^+ = \{v \in S^2 : u \cdot v > 0\}$, $A_u^- = \{v \in S^2 : u \cdot v < 0\}$. Let u_1, u_2, u_3 be distinct points in S^2, and consider the interior $B = \cap_{i=1}^{3} A_{u_i}^+$ of the spherical triangle determined by u_1, u_2, u_3. Let α_i be the angle between u_j, u_k where $\{i,j,k\} = \{1,2,3\}$.

Claim 2.16. $\mu(B) = \sum_{i=1}^{3} \alpha_i - \pi$.

Proof. Let $B_{ij} = A^+_{u_i} \cap A^+_{u_j}$. Then $\mu(B_{ij}) = 2\alpha_k$ where $\{i,j,k\} = \{1,2,3\}$. Let $C = \cup_{1\le i<j\le 3} B_{ij}$. Then $C \cup (-C) = S^2 \setminus \partial B$ is a partition. It follows that $\mu(C) = 2\pi$. On the other hand,

$$\mu(C) = \sum_{1\le i<j\le 3} \mu(B_{ij}) - 2\mu(B) = 2\sum_{i=1}^{3} \alpha_i - 2\mu(B).$$

It follows that $2\pi = 2\sum_{i=1}^{3} \alpha_i - 2\mu(B)$.

□

We recall the cross product in $\mathbb{R}^3$. For $u, v \in \mathbb{R}^3$, let $u \times v$ be the unique element of $\mathbb{R}^3$ such that for $\det(u, v, x) = (u \times v) \cdot x$ for any $x \in \mathbb{R}^3$. Then $u \times v \perp u, v$ and $|u \times v| = |u| \cdot |v| \cdot \sin\alpha$ where α is the angle between u and v.

Claim 2.17.
(i) $(u_1 \times u_2) \times v = -(v \cdot u_2)u_1 + (v \cdot u_1)u_2$.
(ii) $(u_1 \times u_2) \cdot (v_1 \times v_2) = (u_1 \cdot v_1) \cdot (u_2 \cdot v_2) - (u_1 \cdot v_2) \cdot (u_2 \cdot v_1)$.

Let T be a spherical triangle with vertices $u, v, w \in S^2$. Let $d(v, w) = a, d(u, w) = b, d(u, v) = c$ and let α, β, γ be the angles at u, v, w respectively.

Proposition 2.18 (Spherical Cosine Formula)**.**

$$\cos\gamma = \frac{\cos c - \cos a \cos b}{\sin a \sin b}. \tag{48}$$

Proof. We have $|v \times w| = \sin a$, $|u \times w| = \sin b$. It follows that

$$\begin{aligned} \cos c - \cos a \cos b &= u \cdot v - (u \cdot w)(w \cdot v) \\ = (u \times w) \cdot (v \times w) &= |u \times w| \cdot |v \times w| \cdot \cos\gamma \\ &= \sin a \sin b \cos\gamma. \end{aligned} \tag{49}$$

□

Corollary 2.19 (Spherical Pythagoras Theorem)**.** *If* $\gamma = \frac{\pi}{2}$*, then* $\cos c = \cos a \cos b$.

Proposition 2.20 (Spherical Sine Formula)**.** *Let* V *be the volume of the parallelepiped spanned by* u, v, w*. Then*

$$\frac{\sin^2\alpha}{\sin^2 a} = \frac{\sin^2\beta}{\sin^2 b} = \frac{\sin^2\gamma}{\sin^2 c} = \frac{V^2}{\sin^2 a \cdot \sin^2 b \cdot \sin^2 c}. \tag{50}$$

Proof. We may assume that $u = (0, 0, 1)$ and $v = (\sin c, 0, \cos c)$. Then $w = (\sin b \cos\alpha, \sin b \sin\alpha, \cos b)$. It follows that

$$V^2 = \det(u, v, w)^2 = \det \begin{bmatrix} 0 & 0 & 1 \\ \sin c & 0 & \cos c \\ \sin b \cos\alpha & \sin b \sin\alpha & \cos b \end{bmatrix}^2 = \sin^2 c \cdot \sin^2 b \cdot \sin^2\alpha.$$

Similarly

$$V^2 = \sin^2 a \cdot \sin^2 b \cdot \sin^2\gamma = \sin^2 a \cdot \sin^2 c \cdot \sin^2\beta.$$

This implies (50).

□

Define a a function on $d : \mathbb{P}^2 \times \mathbb{P}^2 \to \mathbb{R}$ by

$$d([u], [v]) = \arccos\left(\frac{|u \cdot v|}{|u| \cdot |v|}\right).$$

Proposition 2.21. $d(\cdot, \cdot)$ *defines a metric on* $\mathbb{P}^2$.

3 The Hyperbolic Plane

We first recall the following (abridged) definition of Riemmanian metrics. Let U be an open set in $\mathbb{R}^n$ and let $S(x) = (s_{ij}(x))_{i,j=1}^n$ be a positive definite symmetric matrix, where the $s_{ij} : U \to \mathbb{R}$ are smooth functions. For a smooth curve $\gamma : [a, b] \to U$, define the length of γ with respect to S by

$$L(\gamma) = \int_a^b \left(\dot{\gamma}(t)^T S(\gamma(t))\dot{\gamma}(t)\right)^{\frac{1}{2}} dt. \tag{51}$$

It can be check that $L(\gamma)$ is independent of the parametrization of γ. The riemannian distance determined by $S(x)$ is given by $d(p, q) = \inf_\gamma L(\gamma)$, where γ ranges over all smooth paths $\gamma : [0, 1] \to U$ with $\gamma(0) = p, \gamma(1) = q$. The riemannian volume of $A \subset U$ is given by

$$vol(A) = \int_A \sqrt{\det S(x)}dx.$$

In this section we will study two models of the hyperbolic plane using the relevant riemannian metrics.

3.1 The Upper Half Plane Model

The points of the *Hyperbolic Plane* is the set $\mathbb{H} = \{(x, y) : y > 0\}$. The hyperbolic lines are of two types:

- $\ell = \{(x_0, y) : y > 0\}$ for some $x_0 \in \mathbb{R}$.
- $\ell = \{(x, y) : (x - x_0)^2 + y^2 = r_0^2\ ,\ y > 0\}$ for some $x_0 \in \mathbb{R}$, $0 < r_0 \in \mathbb{R}$.

Claim 3.1. *Any two distinct points in $\mathbb{H}$ are contained in a unique line. For any line ℓ and $p \notin \ell$ there exist infinitely many lines through p that are disjoint from ℓ.*

The metric of the hyperbolic plane is defined by the matrix $S(x, y) = \frac{1}{y^2}I$. Thus the hyperbolic length of $\gamma : [a, b] \to \mathbb{H}$ where $\gamma(t) = (x(t), y(t))$ is given by

$$L_{\mathbb{H}}(\gamma) = \int_a^b \frac{\sqrt{\dot{x}(t)^2 + \dot{y}(t)^2}}{y(t)}dt.$$

The hyperbolic area of $A \subset \mathbb{H}$ is given by

$$\text{area}_{\mathbb{H}}(A) = \int_{(x,y)\in A} \frac{dx\, dy}{y^2}.$$

We will show that the shortest path between $p, q \in \mathbb{H}$ is attained by the hyperbolic segment connecting these two points. For example, if $0 < a < b$ and $p = (x_0, a), q = (x_0, b)$ and $\gamma(t) = (x_0, t)$ for $0 < a \le t \le b$, then

$$d_{\mathbb{H}}(p, q) = L_{\mathbb{H}}(\gamma) = \int_a^b \frac{dt}{t} = \ln \frac{b}{a}.$$

In the sequel, we identify $\mathbb{H}$ with $\{z \in \mathbb{C} : \operatorname{Im} z > 0\}$. Let $SL_2(\mathbb{R}) = \{A \in GL_2(\mathbb{R}) : \det A = 1\}$ and let $PSL_2(\mathbb{R}) = SL_2(\mathbb{R})/\pm I$. $g = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in PSL_2(\mathbb{R})$ acts on $\mathbb{H}$ by fractional linear transformation $g(z) = \frac{az+b}{cz+d}$.

Claim 3.2. *$PSL_2(\mathbb{R})$ transforms lines into lines.*

Claim 3.3. *Let $\gamma : [a, b] \to \mathbb{H}$ be a smooth path and let $g \in PSL_2(\mathbb{R})$. Then $L_{\mathbb{H}}(g(\gamma)) = L_{\mathbb{H}}(\gamma)$.*

Proof. Let $\Gamma(t) = g(\gamma(t))$. Then

$$\Gamma'(t) = \frac{(ad-bc)\gamma'(t)}{(c\gamma(t)+d)^2} = \frac{\gamma'(t)}{(c\gamma(t)+d)^2}.$$

On the other hand

$$\operatorname{Im}\Gamma(t) = \frac{(ad-bc)}{|c\gamma(t)+d|^2} \cdot \operatorname{Im}\gamma(t) = \frac{\operatorname{Im}\gamma(t)}{|c\gamma(t)+d|^2}.$$

It follows that

$$\frac{|\Gamma'(t)|}{\operatorname{Im}\Gamma(t)} = \frac{|\gamma'(t)|}{\operatorname{Im}\gamma(t)}.$$

□

The *hyperbolic distance* $d_{\mathbb{H}}(p,q)$ between two point $p,q \in \mathbb{H}$ is defined as $d_{\mathbb{H}}(z,w) = \inf L_{\mathbb{H}}(\gamma)$ where the infimum is taken over all paths γ in $\mathbb{H}$ with endpoints z, w.

Proposition 3.4.

$$d_{\mathbb{H}}(z,w) = \ln \frac{|z-\overline{w}| + |z-w|}{|z-\overline{w}| - |z-w|}. \tag{52}$$

Proof. Note that both sides are invariant under the action of $PSL_2(\mathbb{R})$. If $z = ip$ and $w = iq$ then (52) can be checked directly. Indeed, suppose $\gamma(t) = x(t) + iy(t) \in \mathbb{H}$ such that $\gamma(0) = ip$ and $\gamma(1) = iq$. Then

$$\begin{aligned} L_{\mathbb{H}}(\gamma) &= \int_0^1 \frac{\sqrt{\dot{x}(t)^2 + \dot{y}(t)^2}}{y(t)} dt \\ &\geq \int_0^1 \frac{|\dot{y}(t)|}{y(t)} dt = |\ln \frac{p}{q}|. \end{aligned} \tag{53}$$

Suppose now that z and w are not on the same vertical line. There is a unique Euclidean semicircle C with radius R and center $\alpha + R$ that contains z and w. Let $g(z) = 1 - \frac{2R}{z-\alpha}$. Then $g(z)$ and $g(w)$ are on the line it and the result follows.

□

Proposition 3.5. *Let T be a hyperbolic triangle with angles α, β, γ. Then*

$$\text{area}_{\mathbb{H}}(T) = \pi - (\alpha + \beta + \gamma). \tag{54}$$

□

We next discuss some aspects of hyperbolic trigonometry. Recall the hyperbolic trigonometric functions $\cosh x = \frac{e^x + e^{-x}}{2}$ and $\sinh x = \frac{e^x - e^{-x}}{2}$.

Claim 3.6. *Let $z, w \in \mathbb{H}$. Then*

$$\cosh d_{\mathbb{H}}(z,w) = 1 + \frac{|z-w|^2}{2 \operatorname{Im} z \operatorname{Im} w}.$$

Proof. By (52)

$$\begin{aligned} \cosh d_{\mathbb{H}}(z,w) &= \frac{1}{2}\left(\frac{|z-\overline{w}| + |z-w|}{|z-\overline{w}| - |z-w|} + \frac{|z-\overline{w}| - |z-w|}{|z-\overline{w}| + |z-w|}\right) \\ \frac{|z-\overline{w}|^2 + |z-w|^2}{|z-\overline{w}|^2 - |z-w|^2} &= 1 + \frac{2(|z-w|^2)}{|z-\overline{w}|^2 - |z-w|^2} \\ &= 1 + \frac{|z-w|^2}{\operatorname{Re}(z\overline{w}) - \operatorname{Re}(zw)} = 1 + \frac{|z-w|^2}{2 \operatorname{Im} z \operatorname{Im} w}. \end{aligned} \tag{55}$$

□

3.2 The Hyperboloid Model

Define the Lorentz bilinear form in $\mathbb{R}^3$ as follows. For $x = (x_0, x_1, x_2), y = (y_0, y_1, y_2) \in \mathbb{R}^3$ let

$$Q(x, y) = -x_0 y_0 + x_1 y_1 + x_2 y_2.$$

Let

$$\tilde{\mathbb{H}} = \{x \in \mathbb{R}^3 : Q(x, x) = -1, x_0 > 0\}.$$

The points of the hyperboloid model are the elements of $\tilde{\mathbb{H}}$, and the lines are nonempty intersections of $\tilde{\mathbb{H}}$ with 2-dimensional linear subspaces of $\mathbb{R}^3$.

Claim 3.7. *Any two distinct points in $\tilde{\mathbb{H}}$ are contained in a unique line. For any line ℓ and $p \notin \ell$ there exist infinitely many lines through p that are disjoint from ℓ.*

Claim 3.8. *If $\gamma : [a, b] \to \tilde{\mathbb{H}}$ is a differentiable curve then $Q(\dot{\gamma}(t), \dot{\gamma}(t)) \geq 0$.*

Proof. Write $\gamma(t) = (x_0(t), x_1(t), x_2(t)) \in \tilde{\mathbb{H}}$. Then $x_0(t) \geq 1$ and $x_0(t)^2 = 1 + x_1(t)^2 + x_2(t)^2$. Differentiating we obtain $x_0(t)\dot{x}_0(t) = x_1(t)\dot{x}_1(t) + x_2(t)\dot{x}_2(t)$. Hence

$$\begin{aligned}
\dot{x}_0(t)^2 &= \frac{(x_1(t)\dot{x}_1(t) + x_2(t)\dot{x}_2(t))^2}{x_0(t)^2} \\
&\leq \frac{x_1(t)^2 + x_2(t)^2}{x_0(t)^2} \cdot \left(\dot{x}_1(t)^2 + \dot{x}_2(t)^2\right) \\
&= \frac{x_0(t)^2 - 1}{x_0(t)^2} \cdot \left(\dot{x}_1(t)^2 + \dot{x}_2(t)^2\right) \\
&\leq \dot{x}_1(t)^2 + \dot{x}_2(t)^2.
\end{aligned}$$

□

Define the length of γ by

$$L_{\tilde{\mathbb{H}}}(\gamma) = \int_a^b \sqrt{Q(\dot{\gamma}(t), \dot{\gamma}(t))}dt.$$

Let

$$\eta = \begin{bmatrix} -1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}.$$

The *Lorentz group* $L = O(1, 2)$ is defined by

$$\begin{aligned}
L &= \{g \in GL_3(\mathbb{R}) : Q(gx, gx) = Q(x, x) \text{ for all } x \in \mathbb{R}^3\} \\
&= \{g \in GL_3(\mathbb{R}) : g^T \eta g = \eta\}.
\end{aligned} \tag{56}$$

The *Proper Lorentz Group* is the connected component of $I \in L$:

$$L^0 = \{g = (g_{ij})_{i,j=0}^2 \in L : g_{00} > 0, \det g = 1\}.$$

Claim 3.9. *All elements of L^0 are bijective self-maps of $\tilde{\mathbb{H}}$.*

Proof. Let $u = (u_0, u_1, u_2) \in \tilde{\mathbb{H}}$ and let $g = (g_{ij}) \in L^0$. Note that $g^T \in L^0$, hence $-g_{00}^2 + g_{10}^2 + g_{20}^2 = -1$. Write $gu = v = (v_0, v_1, v_2)$. Then $g_{00} > 0$ and $-u_0^2 + u_1^2 + u_2^2 = -1$. Hence, by Cauchy-Schwartz

$$\begin{aligned}
v_0 = g_{00}u_0 + g_{10}u_1 + g_{20}u_2 &= \sqrt{1 + g_{10}^2 + g_{20}^2}\sqrt{1 + u_1^2 + u_2^2} + g_{10}u_1 + g_{20}u_2 \\
&\geq \sqrt{1 + g_{10}^2 + g_{20}^2}\sqrt{1 + u_1^2 + u_2^2} - \sqrt{g_{10}^2 + g_{20}^2}\sqrt{u_1^2 + u_2^2} > 0.
\end{aligned} \tag{57}$$

□

We single out two families of elements in L^0. For $\theta \in \mathbb{R}$ let

$$R(\theta) = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix} \in O(2) \quad , \quad \tilde{R}(\theta) = \begin{bmatrix} 1 & 0 \\ 0 & R_\theta \end{bmatrix} \in L^0$$

For $\lambda \in \mathbb{R}$ let

$$L(\lambda) = \begin{bmatrix} \cosh\lambda & -\sinh\lambda \\ -\sinh\lambda & \cosh\lambda \end{bmatrix} \quad , \quad \tilde{L}(\lambda) = \begin{bmatrix} L(\lambda) & 0 \\ 0 & 1 \end{bmatrix} \in L^0$$

Claim 3.10. *L^0 acts transitively on $\tilde{\mathbb{H}}$.*

Proof. Let $u \in \tilde{\mathbb{H}}$. We will show that there exists a $g \in L^0$ such that $gu = (1,0,0)$. Write $u = (\sqrt{r^2+1}, r\cos\theta, r\sin\theta)$. Then $\tilde{R}(-\theta)u = (\sqrt{r^2+1}, r, 0)$. Now let $\lambda = \ln(\sqrt{1+r^2}+r)$. Then $\tilde{L}(\lambda)(\sqrt{r^2+1}, r, 0)^T = (1,0,0)^T$.

□

Corollary 3.11. *For any $u, v \in \tilde{\mathbb{H}}$ there exists a $g \in L^0$ such that $gu = (1,0,0)$ and $gv = (\sqrt{1+\alpha^2}, \alpha, 0)$ for some α.*

Proof. By Claim 3.10 there exists a $g_1 \in L^0$ such that $g_1 u = (1,0,0)^T$. Write $g_1 v = (\sqrt{1+r^2}, r\cos\theta, r\sin\theta)^T$. Then $g = \tilde{R}(-\theta)g_1$ satisfies the required conditions.

□

Claim 3.12. *Let $\gamma : [a,b] \to \tilde{\mathbb{H}}$ be a smooth path and let $g \in L^0$. Then $L_{\tilde{\mathbb{H}}}(g(\gamma)) = L_{\tilde{\mathbb{H}}}(\gamma)$.*

Proof. Let $\gamma(t) = (x_0(t), x_1(t), x_2(t))$ and $\Gamma(t) = g(\gamma(t))$. Then $\dot{\Gamma}(t) = g\dot{\gamma}(t)$ and hence

$$\begin{aligned} L_{\tilde{\mathbb{H}}}(\Gamma) &= \int_a^b \sqrt{Q(\dot{\Gamma}(t), \dot{\Gamma}(t))}dt \\ &= \int_a^b \sqrt{Q(g\dot{\gamma}(t), g\dot{\gamma}(t))}dt \\ &= \int_a^b \sqrt{Q(\dot{\gamma}(t), \dot{\gamma}(t))}dt = L_{\tilde{\mathbb{H}}}(\gamma). \end{aligned} \tag{58}$$

Proposition 3.13. *For $u, v \in \tilde{\mathbb{H}}$*

$$\cosh d_{\tilde{\mathbb{H}}}(u,v) = -Q(u,v).$$

Proof. In view of Claim 3.12 and Corollary 3.11, it suffices to consider the case $u = (1,0,0)$, $v = (\sqrt{1+\alpha^2}, \alpha, 0)$. Let $\gamma : [0,1] \to \tilde{\mathbb{H}}$ such that $\gamma(0) = u$, $\gamma(1) = v$. Write $\gamma(t) = (x_0(t), x_1(t), x_2(t))$. Let $w(t) = \sqrt{x_1(t)^2 + x_2(t)^2}$. Then $w(t)^2 = x_1(t)^2 + x_2(t)^2 = x_0(t)^2 - 1$. Therefore

$$\begin{aligned} & w(t)|\dot{w}(t)| = |\dot{x}_1(t)x_1(t) + \dot{x}_2(t)x_2(t)| \\ & \le \sqrt{\dot{x}_1(t)^2 + \dot{x}_1(t)^2} \cdot \sqrt{x_1(t)^2 + x_1(t)^2} \\ & = w(t)\sqrt{\dot{x}_1(t)^2 + \dot{x}_1(t)^2}. \end{aligned} \tag{59}$$

Hence

$$\dot{x}_1(t)^2 + \dot{x}_1(t)^2 \ge \dot{w}(t)^2. \tag{60}$$

Moreover $x_0(t)\dot{x}_0(t) = w(t)\dot{w}(t)$, and hence

$$\dot{x}_0(t)^2 = \frac{w(t)^2\dot{w}(t)^2}{x_0(t)^2} = \frac{w(t)^2\dot{w}(t)^2}{w(t)^2+1}. \tag{61}$$

Combining (60) and (60) we obtain

$$\begin{aligned} L_{\tilde{\mathbb{H}}}(\gamma) &= \int_0^1 \left(-\dot{x}_0(t)^2 + \dot{x}_1(t)^2 + \dot{x}_2(t)^2\right)^{\frac{1}{2}} dt \\ &\geq \int_0^1 \left(\dot{w}(t)^2 - \dot{x}_0(t)^2\right)^{\frac{1}{2}} dt \\ &= \int_0^1 \left(\dot{w}(t)^2 - \frac{w(t)^2\dot{w}(t)^2}{w(t)^2+1}\right)^{\frac{1}{2}} dt \\ &= \int_0^1 \frac{\dot{w}(t)dt}{\sqrt{w(t)^2+1}} \\ &= \ln\left(w(t) + \sqrt{w(t)^2+1}\right)|_{t=0}^{t=1} \\ &= \ln(\alpha + \sqrt{\alpha^2+1}). \end{aligned} \tag{62}$$

Therefore

$$\begin{aligned} \cosh L_{\tilde{\mathbb{H}}}(\gamma) &\geq \frac{1}{2}\left(\alpha + \sqrt{\alpha^2+1} + \frac{1}{\alpha+\sqrt{\alpha^2+1}}\right) \\ &= \sqrt{\alpha^2+1} = -Q(u,v). \end{aligned} \tag{63}$$

On the other hand, taking $\gamma(t) = (\sqrt{1+t^2}, t, 0)$ for $0 \leq t \leq \alpha$, we get

$$\begin{aligned} \cosh L_{\tilde{\mathbb{H}}}(\gamma) &= \cosh\left(\int_0^\alpha \frac{dt}{\sqrt{1+t^2}}\right) = \cosh\left(\ln(\alpha+\sqrt{\alpha^2+1})\right) \\ &= \sqrt{\alpha^2+1} = -Q(u,v). \end{aligned}$$

□

Proposition 3.14. *Let $u, v \in \tilde{\mathbb{H}}$ and let $d = d_{\tilde{\mathbb{H}}}(u,v)$. Let $z = \frac{v-\cosh(d)u}{\sinh(d)}$. Then $\gamma : [0,d] \to \tilde{\mathbb{H}}$ given by $\gamma(t) = \cosh(t)u + \sinh(t)z$ is a geodesic between u and v.*

Proof. First note that

$$\begin{aligned} Q(z,z) &= \left(\frac{v-\cosh(d)u}{\sinh(d)}, \frac{v-\cosh(d)u}{\sinh(d)}\right) \\ &= \frac{Q(v-\cosh(d)u, v-\cosh(d)u)}{\sinh^2(d)} \\ &= \frac{Q(u,u) + \cosh^2(d)Q(u,u) - 2\cosh(d)Q(u,v)}{\sinh^2(d)} \\ &= \frac{-1-\cosh^2 d + 2\cosh^2 d}{\sinh^2(d)} = \frac{\sinh^2(d)}{\sinh^2(d)} = 1. \end{aligned} \tag{64}$$

Next observe that $\dot{\gamma}(t) = \sinh(t)u + \cosh(t)z$, and that

$$\begin{aligned} Q(u,z) &= Q\left(u, \frac{v-\cosh(d)u}{\sinh(d)}\right) = \frac{Q(u,v) - \cosh(d)Q(u,u)}{\sinh^2(d)} \\ &= \frac{-\cosh(d)+\cosh(d)}{\sinh^2(d)} = 0. \end{aligned} \tag{65}$$

Hence

$$\begin{aligned} Q\left(\gamma(t),\gamma(t)\right) &= Q\left(\cosh(t)u + \sinh(t)z, \cosh(t)u + \sinh(t)z\right) \\ &= \cosh^2(t)Q(u,u) + \sinh^2(t)Q(z,z) + \cosh(t)\sinh(t)Q(u,z) \\ &= -\cosh^2(t) + \sinh^2(t) = -1. \end{aligned} \tag{66}$$

It follows that $\gamma(t) \in \tilde{\mathbb{H}}$. Furthermore

$$\begin{aligned} Q\left(\dot{\gamma}(t), \dot{\gamma}(t)\right) &= Q(\sinh(t)u + \cosh(t)z, \sinh(t)u + \cosh(t)z) \\ &= \sinh^2(t)Q(u,u) + \cosh^2(t)Q(z,z) \\ &= -\sinh^2(t) + \cosh^2(t) = 1. \end{aligned} \tag{67}$$

Therefore

$$\int_{t=0}^{d} \sqrt{Q\left(\dot{\gamma}(t), \dot{\gamma}(t)\right)}dt = d,$$

and hence γ is a geodesic between u and v.

□

3.3 Hyperbolic Trigonometry

The *hyperbolic cross product* $u \times_Q v$ of $u, v \in \mathbb{R}^3$ is defined as the unique element of $\mathbb{R}^3$ such that $\det(u, v, w) = Q(u \times_Q v, w)$. Defining $R : \mathbb{R}^3 \to \mathbb{R}^3$ by $R(u_1, u_2, u_3) = (-u_1, u_2, u_3)$, we have $Q(u, v) = (Ru) \cdot v$ and $u \times_Q v = R(u \times v) = -(Ru_1) \times (Ru_2)$. Define the hyperbolic norm of an element $u \in \mathbb{R}^3$ by

$$|u|_Q = \left\{ \begin{array}{ll} \sqrt{Q(u,u)} & Q(u,u) \geq 0, \\ i\sqrt{-Q(u,u)} & Q(u,u) < 0. \end{array} \right.$$

For example, if $u \in \tilde{\mathbb{H}}$ then $|u|_Q = i$.

Corollary 3.15. *Let $u, v \in \tilde{\mathbb{H}}$. Then*

$$|u \times_Q v|_Q = \sinh d_{\tilde{\mathbb{H}}}(u, v). \tag{68}$$

Proof.

$$Q(u \times_Q v, u \times_Q v) = Q(u,v)^2 - Q(u,u) \cdot Q(v,v) = \cosh^2 d_{\tilde{\mathbb{H}}}(u,v) - 1 = \sinh^2 d_{\tilde{\mathbb{H}}}(u,v).$$

As $d_{\tilde{\mathbb{H}}}(u, v) \geq 0$, it follows that $Q(u \times_Q v, u \times_Q v) \geq 0$ and $\sinh d_{\tilde{\mathbb{H}}}(u, v) \geq 0$. Hence

$$|u \times_Q v|_Q = \sqrt{Q(u \times_Q v, u \times_Q v)} = \sinh d_{\tilde{\mathbb{H}}}(u, v).$$

□

The hyperbolic counterpart of Claim 2.17 is the following

Claim 3.16.
(i) $(u_1 \times_Q u_2) \times_Q v = Q(u_2, v)u_1 - Q(u_1, v)u_2$.
(ii) $Q(u_1 \times_Q u_2, v_1 \times_Q v_2) = Q(u_1, v_2)Q(u_2, v_1) - Q(u_1, v_1)Q(u_2, v_2)$.

Proof. (i)

$$\begin{aligned} (u_1 \times_Q u_2) \times_Q v &= -R(u_1 \times_Q u_2) \times Rv = -(u_1 \times u_2) \times Rv \\ &= (Rv \cdot u_2)u_1 - (Rv \cdot u_1)u_2 \\ &= Q(u_2, v)u_1 - Q(u_1, v)u_2. \end{aligned}$$

(ii)

$$\begin{aligned} Q(u_1 \times_Q u_2, v_1 \times_Q v_2) &= R(u_1 \times_Q u_2) \cdot (v_1 \times_Q v_2) \\ &= (u_1 \times u_2) \cdot (v_1 \times_Q v_2) = -(u_1 \times u_2) \cdot (Rv_1 \times Rv_2) \\ &= -(u_1 \cdot Rv_1)(u_2 \cdot Rv_2) + (u_1 \cdot Rv_2)(u_2 \cdot Rv_1) \\ &= Q(u_1, v_2)Q(u_2, v_1) - Q(u_1, v_1)Q(u_2, v_2). \end{aligned}$$

□

Let T be a hyperbolic triangle with vertices $u, v, w \in \tilde{\mathbb{H}}$. Let $d_{\tilde{\mathbb{H}}}(v, w) = a, d_{\tilde{\mathbb{H}}}(u, w) = b, d_{\tilde{\mathbb{H}}}(u, v) = c$ and let α, β, γ be the angles at u, v, w respectively.

Claim 3.17.

$$\cos\gamma = \frac{Q(u \times_Q w, v \times_Q w)}{|u \times_Q w| \cdot |v \times_Q w|}. \tag{69}$$

Proposition 3.18 (Hyperbolic Cosine Formula)**.**

$$\cos\gamma = \frac{\cosh a \cosh b - \cosh c}{\sinh a \sinh b}. \tag{70}$$

Proof. By Proposition 3.13, $\cosh a = -Q(v, w)$, $\cosh b = -Q(u, w)$ and $\cosh c = -Q(u, v)$. By Corollary 3.15

$$|u \times_Q w|_Q = \sinh d_{\tilde{\mathbb{H}}}(u, w) = \sinh b$$

and

$$|v \times_Q w|_Q = \sinh d_{\tilde{\mathbb{H}}}(v, w) = \sinh a.$$

Substituting in (3.17) we obtain

$$\begin{aligned}\cos\gamma &= \frac{Q(u \times_Q w, v \times_Q w)}{|u \times_Q w| \cdot |v \times_Q w|} \\ &= \frac{Q(u, w)Q(v, w) - Q(u, v)Q(w, w)}{\sinh a \sinh b} \\ &= \frac{\cosh a \cosh b - \cosh c}{\sinh a \sinh b}.\end{aligned} \tag{71}$$

□

Proposition 3.19. *For any $0 < \alpha < \frac{(n-2)\pi}{n}$ there exists a regular n-gon in $\tilde{\mathbb{H}}$ with all angles equal to α.*

Proof. For $x > 0$ and $0 \le k \le n - 1$ let

$$u_{k,n}(x) = \left(\cosh x, \sinh x \cos\frac{2\pi k}{n}, \sinh x \sin\frac{2\pi k}{n}\right).$$

Let $P_n(x)$ denote the n-gon with vertices $\{u_{k,n}(x)\}_{k=0}^{n-1}$. Let $d_n(x)$ denote the edge length of $P_n(x)$. Then

$$\begin{aligned}\cosh d_n(x) &= -Q\left(u_{0,n}(x), u_{1,n}(x)\right) \\ &= -Q\left((\cosh x, \sinh x, 0), (\cosh x, \sinh x \cos\frac{2\pi}{n}, \sinh x \sin\frac{2\pi}{n})\right) \\ &= \cosh^2 x - \sinh^2 x \cos\frac{2\pi}{n} \\ &= 1 + \sinh^2 x(1 - \cos\frac{2\pi}{n}).\end{aligned} \tag{72}$$

Let $e_n(x) = d_{\tilde{\mathbb{H}}}(u_0(x), u_2(x))$. Then

$$\begin{aligned}\cosh e_n(x) &= -Q\left(u_{0,n}(x), u_{2,n}(x)\right) \\ &= -Q\left((\cosh x, \sinh x, 0), (\cosh x, \sinh x \cos\frac{4\pi}{n}, \sinh x \sin\frac{2\pi}{n})\right) \\ &= \cosh^2 x - \sinh^2 x \cos\frac{4\pi}{n} \\ &= 1 + \sinh^2 x(1 - \cos\frac{4\pi}{n}).\end{aligned} \tag{73}$$

Let $\gamma_n(x)$ denote the angle of $P_n(x)$. By the hyperbolic cosine theorem

$$\begin{aligned}\cos\gamma_n(x) &= \frac{\cosh^2 d_n(x) - \cosh e_n(x)}{\sinh^2 d_n(x)} \\ &= \frac{\big(1+\sinh^2 x(1-\cos\frac{2\pi}{n})\big)^2 - \big(1+\sinh^2 x(1-\cos\frac{4\pi}{n})\big)}{\big(1+\sinh^2 x(1-\cos\frac{2\pi}{n})\big)^2 - 1}.\end{aligned} \tag{74}$$

Therefore

$$\begin{aligned}\lim_{x\to 0}\cos\gamma_n(x) &= \lim_{x\to 0}\frac{\big(1+\sinh^2 x(1-\cos\frac{2\pi}{n})\big)^2 - \big(1+\sinh^2 x(1-\cos\frac{4\pi}{n})\big)}{\big(1+\sinh^2 x(1-\cos\frac{2\pi}{n})\big)^2 - 1} \\ &= \lim_{t\to 0}\frac{\big(1+2t(1-\cos\frac{2\pi}{n})\big) - \big(1+t(1-\cos\frac{4\pi}{n})\big)}{\big(1+2t(1-\cos\frac{2\pi}{n})\big) - 1} \\ &= \frac{1-2\cos\frac{2\pi}{n}+\cos\frac{4\pi}{n}}{2(1-\cos\frac{2\pi}{n})} \\ &= \frac{\cos^2\frac{2\pi}{n}-\cos\frac{2\pi}{n}}{1-\cos\frac{2\pi}{n}} \\ &= -\cos\frac{2\pi}{n} = \cos\frac{(n-2)\pi}{n}.\end{aligned} \tag{75}$$

It follows $\lim_{x\to 0}\gamma_n(x) = \frac{(n-2)\pi}{n}$. On the other hand

$$\begin{aligned}\lim_{x\to\infty}\cos\gamma_n(x) &= \lim_{x\to\infty}\frac{\big(1+\sinh^2 x(1-\cos\frac{2\pi}{n})\big)^2 - \big(1+\sinh^2 x(1-\cos\frac{4\pi}{n})\big)}{\big(1+\sinh^2 x(1-\cos\frac{2\pi}{n})\big)^2 - 1} \\ &= \lim_{t\to\infty}\frac{\big(1+t(1-\cos\frac{2\pi}{n})\big)^2 - \big(1+t(1-\cos\frac{4\pi}{n})\big)}{\big(1+t(1-\cos\frac{2\pi}{n})\big)^2 - 1} = 1.\end{aligned} \tag{76}$$

It follows $\lim_{x\to\infty}\gamma_n(x) = 0$. By continuity, for any $0 < \alpha < \frac{(n-2)\pi}{n}$ there exists an x such that $\gamma_n(x) = \alpha$.

$\square$