Betti Numbers of Complexes with Highly Connected Links

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Abstract

Let $\Delta_{n-1}^{(k)}$ denote the k-dimensional skeleton of the $(n-1)$-simplex Δ_{n-1} and consider a complex $\Delta_{n-1}^{(k-1)} \subset X \subset \Delta_{n-1}^{(k)}$. Let $\mathbb{K}$ be a field and let $0 \leq \ell < k$. It is shown that if $\tilde{H}_{k-\ell-2}(\mathrm{lk}(X,\tau);\mathbb{K}) = 0$ for all ℓ-dimensional faces τ of X then

$$\dim \tilde{H}_{k-1}(X;\mathbb{K}) \leq \frac{\binom{n-1}{\ell}\binom{n-\ell-2}{k-\ell}}{\binom{k+1}{\ell+1}}$$

with equality iff $\mathrm{lk}(X,\tau)$ is a $(k-\ell-1)$-hypertree for all ℓ-dimensional simplices τ of Δ_{n-1}. Examples based on sum complexes show that the bound is asymptotically tight for all fixed k, ℓ as $n \to \infty$.

1 Introduction

Let X be a simplicial complex on the vertex set V. Numerous problems in topological combinatorics ask for estimates on some global invariants of X, e.g. its connectivity or Betti numbers, given that X satisfies certain local properties. One remarkable local to global result of this nature is Garland's theorem [3]. We first recall some definitions. The induced subcomplex of X on $V' \subset V$ is $X[V'] = \{\sigma \in X : \sigma \subset V'\}$. Denote the star, link and costar of a simplex $\tau \in X$ by

$$\begin{aligned} \mathrm{st}(X,\tau) &= \{\sigma \in X : \sigma \cup \tau \in X\} \\ \mathrm{lk}(X,\tau) &= \{\sigma \in \mathrm{st}(X,\tau) : \sigma \cap \tau = \emptyset\} \\ \mathrm{cost}(X,\tau) &= \{\sigma \in X : \sigma \not\supset \tau\}. \end{aligned}$$

Let $X^{(j)}$ denote the j-th skeleton of X and let $X(j)$ be the family of j-dimensional simplices of X. Denote $f_j(X) = |X(j)|$. Assume that X is a pure k-dimensional complex, and define a positive weight function on its simplices by

$$c(\sigma) = (k - \dim \sigma)!|\{\tau \in X(k) : \tau \supset \sigma\}|.$$

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For $\tau \in X$ let c_τ be the induced weight function on $\mathrm{lk}(X,\tau)$ given by $c_\tau(\alpha) = c(\tau \cup \alpha)$. For $-1 \leq j \leq k$ let $C^j(X;\mathbb{R})$ denote the space of real valued j-cochains of X and let $d_j : C^j(X;\mathbb{R}) \to C^{j+1}(X;\mathbb{R})$ denote the j-th coboundary map of X. Let $d_j^* : C^{j+1}(X;\mathbb{R}) \to C^j(X;\mathbb{R})$ be the adjoint of d_j with respect to the weight function c. Let $L_j = d_{j-1}d_{j-1}^* + d_j^* d_j : C^j(X;\mathbb{R}) \to C^j(X;\mathbb{R})$ be the j-th Laplacian of X and let $\mu_j(X)$ denote its minimal eigenvalue.

Theorem 1 (Garland [3])**.** *Let $-1 \leq \ell < k-1$. If $\mu_{k-\ell-2}(\mathrm{lk}(X,\tau)) > \frac{\ell+1}{k}$ for all $\tau \in X(\ell)$, then $\tilde{H}_{k-1}(X;\mathbb{R}) = 0$.*

Garland's Theorem and its variants have applications in a wide range of areas including representation theory, geometric group theory, hypergraph matching and random complexes (see e.g. [3, 2, 1, 4]). Here we study the following question that naturally arises in connection with Theorem 1: What can be said concerning $\tilde{H}_{k-1}(X)$ if, instead of $\mu_{k-\ell-2}(\mathrm{lk}(X,\tau)) > \frac{\ell+1}{k}$, it is only assumed that $\tilde{H}_{k-\ell-2}(\mathrm{lk}(X,\tau)) = 0$ for all $\tau \in X(\ell)$?

Let $\mathbb{K}$ be a fixed field. Let $\tilde{H}_j(X) = \tilde{H}_j(X;\mathbb{K})$ and $\tilde{\beta}_j(X) = \dim_{\mathbb{K}} \tilde{H}_j(X;\mathbb{K})$ be the reduced homology groups and reduced Betti numbers of X over $\mathbb{K}$. For $0 \leq \ell < k$ and $-1 \leq j$ let

$$\lambda_{\ell,j}(X) = \sum_{\tau \in X(\ell)} \tilde{\beta}_j(\mathrm{lk}(X,\tau)).$$

Let

$$B_{n,k,\ell} = \frac{\binom{n-1}{\ell}\binom{n-\ell-2}{k-\ell}}{\binom{k+1}{\ell+1}}$$

and

$$F_{n,k,\ell} = \binom{n-1}{k} - B_{n,k,\ell} = \frac{\binom{n}{\ell+1}\binom{n-\ell-2}{k-\ell-1}}{\binom{k+1}{\ell+1}}.$$

Let $\Delta(V)$ denote the simplex on the vertex set V. Let $[n] = \{0,\ldots,n-1\}$ and let $\Delta_{n-1} = \Delta([n])$ be the $(n-1)$-simplex on $[n]$.

Theorem 2. *If $\Delta_{n-1}^{(k-1)} \subset X \subset \Delta_{n-1}$ and $0 \leq \ell < k$ then*

$$\tilde{\beta}_{k-1}(X) \leq \frac{1}{\binom{k+1}{\ell+1}} \lambda_{\ell,k-\ell-2}(X) + B_{n,k,\ell}. \tag{1}$$

Theorem 2 implies the following

Corollary 3. *Suppose $\Delta_{n-1}^{(k-1)} \subset X \subset \Delta_{n-1}$ satisfies $\lambda_{\ell,k-\ell-2}(X) = 0$. Then*

$$\tilde{\beta}_{k-1}(X) \leq B_{n,k,\ell}. \tag{2}$$

The next three results concern some aspects of the equality cases in Corollary 3. A complex $\Delta(V)^{(r-1)} \subset Y \subset \Delta(V)^{(r)}$ is an *r-hypertree over* $\mathbb{K}$ on the vertex set V (abbreviated r-hypertree for a fixed field $\mathbb{K}$) if $\tilde{H}_*(Y;\mathbb{K}) = 0$. It is easy to check that Y is an r-hypertree iff $f_r(Y) = \binom{|V|-1}{r}$ and either $\tilde{H}_{r-1}(Y;\mathbb{K}) = 0$ or $\tilde{H}_r(Y;\mathbb{K}) = 0$. See Kalai's paper [5] for further discussion, including a Cayley type formula for the weighted enumeration of rational hypertrees.

Theorem 4. *Let $0 \le \ell < k$ and suppose $\Delta_{n-1}^{(k-1)} \subset X \subset \Delta_{n-1}^{(k)}$ satisfies $\lambda_{\ell,k-\ell-2}(X) = 0$. Then the following three conditions are equivalent.*

(a) $\tilde{\beta}_{k-1}(X) = B_{n,k,\ell}$.

(b) $\tilde{\beta}_k(X) = 0$ *and* $f_k(X) = F_{n,k,\ell}$.

(c) $\mathrm{lk}(X, \tau)$ *is a* $(k - \ell - 1)$*-hypertree on* $[n] - \tau$ *for all* $\tau \in \Delta_{n-1}(\ell)$.

The next result asserts that the bound (2) is asymptotically tight for fixed k, ℓ and $n \to \infty$.

Theorem 5. *Let $0 \le \ell < k$ be fixed. Then for any prime number $n > k$ there exists a complex $\Delta_{n-1}^{(k-1)} \subset X_{n,k,\ell} \subset \Delta_{n-1}^{(k)}$ such that $\lambda_{\ell,k-\ell-2}(X_{n,k,\ell}) = 0$ and*

$$\tilde{\beta}_{k-1}(X_{n,k,\ell}) \ge (1 - O(n^{-1}))B_{n,k,\ell}.$$

Finally, we give examples that show the optimality of (2) for $\ell = 0$ and $k \le 3$.

Theorem 6. *Let $1 \le k \le 3$. Then for infinitely many n's there exist complexes $\Delta_{n-1}^{(k-1)} \subset J_{n,k} \subset \Delta_{n-1}^{(k)}$ such that $\tilde{\beta}_{k-2}(\mathrm{lk}(J_{n,k}, v)) = 0$ for all $v \in \Delta_{n-1}(0)$ and*

$$\tilde{\beta}_{k-1}(J_{n,k}) = B_{n,k,0} = \frac{1}{k+1}\binom{n-2}{k}.$$

The paper is organized as follows: In Section 2 we prove a monotonicity result (Proposition 7) that directly implies Theorem 2. The characterization of equality cases (Theorem 4) is established in Section 3. In Section 4 we recall the notion of sum complexes and prove an upper bound on the Betti number of their links (Proposition 11). This result is the main ingredient in the proof of Theorem 5 given in Section 5. In Section 6 we describe the constructions that yield Theorem 6. We conclude in Section 7 with some remarks and open problems.

2 The Upper Bound

The main ingredient in the proof of Theorem 2 is the following monotonicity result.

Proposition 7. *Let $0 \le \ell < k$. If $X \subset \Delta_{n-1}^{(k)}$ and $\sigma \in X(k)$ then*

$$\begin{aligned} &\lambda_{\ell,k-\ell-2}(X - \sigma) - \lambda_{\ell,k-\ell-2}(X) \\ &\le \binom{k+1}{\ell+1}\left(\tilde{\beta}_{k-1}(X - \sigma) - \tilde{\beta}_{k-1}(X)\right). \end{aligned} \tag{3}$$

Proof: First note that

$$\tilde{\beta}_{k-1}(X) \le \tilde{\beta}_{k-1}(X - \sigma) \le \tilde{\beta}_{k-1}(X) + 1.$$

Let $\tau \in X(\ell)$. If $\tau \subset \sigma$ then

$$\tilde{\beta}_{k-\ell-2}(\mathrm{lk}(X, \tau)) \le \tilde{\beta}_{k-\ell-2}(\mathrm{lk}(X - \sigma, \tau)) \le \tilde{\beta}_{k-\ell-2}(\mathrm{lk}(X, \tau)) + 1. \tag{4}$$

On the other hand, if $\tau \not\subset \sigma$ then $\mathrm{lk}(X,\tau) = \mathrm{lk}(X-\sigma,\tau)$. Summing (4) over all $\tau \in X(\ell)$ we obtain

$$\lambda_{\ell,k-\ell-2}(X-\sigma) \le \lambda_{k-\ell-2}(X) + \binom{k+1}{\ell+1} \tag{5}$$

Consider two cases:
(i) $\tilde{\beta}_{k-1}(X-\sigma) = \tilde{\beta}_{k-1}(X)+1$. Then by (5)

$$\binom{k+1}{\ell+1}\left(\tilde{\beta}_{k-1}(X-\sigma) - \tilde{\beta}_{k-1}(X)\right)$$
$$= \binom{k+1}{\ell+1} \ge \lambda_{\ell,k-\ell-2}(X-\sigma) - \lambda_{k-\ell-2}(X).$$

Thus (3) holds.
(ii) $\tilde{\beta}_{k-1}(X-\sigma) = \tilde{\beta}_{k-1}(X)$. To establish (3) it suffices to show that $\lambda_{\ell,k-\ell-2}(X-\sigma) = \lambda_{\ell,k-\ell-2}(X)$, or equivalently that if $\tau \in \sigma(\ell)$ then

$$\tilde{\beta}_{k-\ell-2}(\mathrm{lk}(X,\tau)) = \tilde{\beta}_{k-\ell-2}(\mathrm{lk}(X-\sigma,\tau)). \tag{6}$$

Consider the decompositions

$$X-\sigma = \mathrm{cost}(X,\tau) \cup \mathrm{st}(X-\sigma,\tau)$$

and

$$X = \mathrm{cost}(X,\tau) \cup \mathrm{st}(X,\tau).$$

Then

$$\mathrm{cost}(X,\tau) \cap \mathrm{st}(X-\sigma,\tau) = \partial\tau * \mathrm{lk}(X-\sigma,\tau)$$

and

$$\mathrm{cost}(X,\tau) \cap \mathrm{st}(X,\tau) = \partial\tau * \mathrm{lk}(X,\tau).$$

Note that $\tilde{H}_{k-2}(\partial\tau * Y) \cong \tilde{H}_{k-\ell-2}(Y)$ for any Y. Hence by Mayer-Vietoris we obtain a commutative diagram

$$\begin{array}{ccccccccc}
\tilde{H}_{k-1}(\mathrm{cost}(X,\tau)) & \longrightarrow & \tilde{H}_{k-1}(X-\sigma) & \longrightarrow & \tilde{H}_{k-\ell-2}(\mathrm{lk}(X-\sigma,\tau)) & \longrightarrow & \tilde{H}_{k-2}(\mathrm{cost}(X,\tau)) & \longrightarrow & \tilde{H}_{k-2}(X-\sigma) \\
\downarrow (i_1)_* & & \downarrow (i_2)_* & & \downarrow (i_3)_* & & \downarrow (i_4)_* & & \downarrow (i_5)_* \\
\tilde{H}_{k-1}(\mathrm{cost}(X,\tau)) & \longrightarrow & \tilde{H}_{k-1}(X) & \longrightarrow & \tilde{H}_{k-\ell-2}(\mathrm{lk}(X,\tau)) & \longrightarrow & \tilde{H}_{k-2}(\mathrm{cost}(X,\tau)) & \longrightarrow & \tilde{H}_{k-2}(X)
\end{array}$$

where the rows are exact and the i_*'s are induced by inclusion maps. Clearly $(i_1)_*$ and $(i_4)_*$ are the identity maps. As the removal of the k-dimensional simplex σ does not effect the $(k-2)$-homology, it follows that $(i_5)_*$ is an isomorphism. The assumption $\tilde{\beta}_{k-1}(X-\sigma) = \tilde{\beta}_{k-1}(X)$ implies that $(i_2)_*$ is an isomorphism. It follows by the 5-lemma that $(i_3)_*$ is an isomorphism as well, and thus (6) holds. This completes the proof of (3).

$\square$

Proof of Theorem 2: First note that if $\tau \in \Delta_{n-1}(\ell)$ and $\ell < k$ then $\mathrm{lk}(\Delta_{n-1}^{(k-1)},\tau) \cong \Delta_{n-\ell-2}^{(k-\ell-2)}$ and that $\tilde{\beta}_j(\Delta_m^{(j)}) = \binom{m}{j+1}$. Secondly, as both $\tilde{H}_{k-1}(X)$ and $\tilde{H}_{k-\ell-2}(\mathrm{lk}(X,\tau))$ for $\tau \in \Delta_{n-1}(\ell)$

depend only on the k-dimensional skeleton of X, we may assume that $X \subset \Delta_{n-1}^{(k)}$. By repeatedly removing k-simplicies from X and using (3) it follows that

$$\begin{aligned}
&\binom{k+1}{\ell+1}\tilde{\beta}_{k-1}(X) - \lambda_{\ell,k-\ell-2}(X) \\
&\leq \binom{k+1}{\ell+1}\tilde{\beta}_{k-1}(\Delta_{n-1}^{(k-1)}) - \lambda_{\ell,k-\ell-2}(\Delta_{n-1}^{(k-1)}) \\
&= \binom{k+1}{\ell+1}\binom{n-1}{k} - \binom{n}{\ell+1}\binom{n-\ell-2}{k-\ell-1} \\
&= \binom{n-1}{\ell}\binom{n-\ell-2}{k-\ell}.
\end{aligned}$$

□

Theorem 2 can be also formulated as the following upper bound on $\tilde{\beta}_k(X)$.

Theorem 8. *Let $X \subset \Delta_{n-1}^{(k)}$. Then for any $-1 \leq \ell < k$*

$$\binom{k+1}{\ell+1}\tilde{\beta}_k(X) \leq \sum_{\tau \in X(\ell)} \tilde{\beta}_{k-\ell-1}(\mathrm{lk}(X,\tau)). \tag{7}$$

Proof: As both sides of (7) do not depend on the $(k-1)$-skeleton of X, we may assume that $X \supset \Delta_{n-1}^{(k-1)}$. The exact sequence for the pair $(X, \Delta_{n-1}^{(k-1)})$

$$0 \to \tilde{H}_k(X) \to H_k(X, \Delta_{n-1}^{(k-1)}) \to \tilde{H}_{k-1}(\Delta_{n-1}^{(k-1)}) \to \tilde{H}_{k-1}(X) \to 0$$

implies that

$$\tilde{\beta}_k(X) = \tilde{\beta}_{k-1}(X) + f_k(X) - \binom{n-1}{k}. \tag{8}$$

Similarly, for each $\tau \in \Delta_{n-1}(\ell)$

$$\tilde{\beta}_{k-\ell-1}(\mathrm{lk}(X,\tau)) = \tilde{\beta}_{k-\ell-2}(\mathrm{lk}(X,\tau)) + f_{k-\ell-1}(\mathrm{lk}(X,\tau)) - \binom{n-\ell-2}{k-\ell-1}. \tag{9}$$

Summing (9) over all $\tau \in \Delta_{n-1}(\ell)$ we obtain

$$\begin{aligned}
\lambda_{\ell,k-\ell-1}(X) &= \lambda_{\ell,k-\ell-2}(X) + \sum_{\tau \in \Delta_{n-1}(\ell)} f_{k-\ell-1}(\mathrm{lk}(X,\tau)) - \binom{n}{\ell+1}\binom{n-\ell-2}{k-\ell-1} \\
&= \lambda_{\ell,k-\ell-2}(X) + \binom{k+1}{\ell+1}\big(f_k(X) - F_{n,k,\ell}\big).
\end{aligned} \tag{10}$$

Combining (8), (1) and (10) it follows that

$$\begin{aligned}
\tilde{\beta}_k(X) &= \tilde{\beta}_{k-1}(X) + f_k(X) - \binom{n-1}{k} \\
&\leq \frac{1}{\binom{k+1}{\ell+1}}\lambda_{\ell,k-\ell-2}(X) + B_{n,k,\ell} + f_k(X) - \binom{n-1}{k} \\
&= \frac{1}{\binom{k+1}{\ell+1}}\Big(\lambda_{\ell,k-\ell-1}(X) - \binom{k+1}{\ell+1}\big(f_k(X) - F_{n,k,\ell}\big)\Big) + B_{n,k,\ell} + f_k(X) - \binom{n-1}{k} \\
&= \frac{1}{\binom{k+1}{\ell+1}}\lambda_{\ell,k-\ell-1}(X).
\end{aligned}$$

□

3 Characterizations of Equality

In this section we prove Theorem 4. Denote the support of a k-chain $z = \sum_{\sigma \in Y(k)} a_\sigma \sigma$ of a complex Y by $\mathrm{supp}(z) = \{\sigma \in Y(k) : a_\sigma \neq 0\}$. We shall need the following observation.

Claim 9. *Let $Y \subset \Delta_{n-1}^{(k)}$ and let $0 \neq z \in \tilde{H}_k(Y)$. If $\sigma \in \mathrm{supp}(z)$ and $\tau \in \sigma(\ell)$ then $\tilde{\beta}_{k-\ell-1}(\mathrm{lk}(Y,\tau)) > 0$.*

Proof: The assumptions imply that $\tilde{\beta}_k(\mathrm{cost}(Y,\tau)) < \tilde{\beta}_k(Y)$. Using, as in the proof of Proposition 7, the exact sequence

$$\tilde{H}_k(\mathrm{cost}(Y,\tau)) \to \tilde{H}_k(Y) \to \tilde{H}_{k-\ell-1}(\mathrm{lk}(Y,\tau))$$

it follows that

$$\tilde{\beta}_{k-\ell-1}(\mathrm{lk}(Y,\tau)) \geq \tilde{\beta}_k(Y) - \tilde{\beta}_k(\mathrm{cost}(Y,\tau)) > 0.$$

□

Proof of Theorem 4: (a) $\Rightarrow$ (b): We first show that (a) implies $\tilde{\beta}_k(X) = 0$. Otherwise choose an inclusion-wise minimal $\Delta_{n-1}^{(k-1)} \subset X \subset \Delta_{n-1}^{(k)}$ such that both $\lambda_{\ell,k-\ell-2}(X) = 0$ and $\tilde{\beta}_{k-1}(X) = B_{n,k,\ell}$, but $\tilde{\beta}_k(X) > 0$. Let $0 \neq z \in \tilde{H}_k(X)$ and let $\sigma \in \mathrm{supp}(z)$. Then $\tilde{\beta}_k(X - \sigma) = \tilde{\beta}_k(X) - 1$ and hence by (8)

$$\tilde{\beta}_{k-1}(X - \sigma) = \tilde{\beta}_{k-1}(X) = B_{n,k,\ell}. \tag{11}$$

Proposition 7 thus implies

$$\begin{aligned}\binom{k+1}{\ell+1} B_{n,k,\ell} &= \binom{k+1}{\ell+1}\tilde{\beta}_{k-1}(X) - \lambda_{\ell,k-\ell-2}(X) \\ &\leq \binom{k+1}{\ell+1}\tilde{\beta}_{k-1}(X-\sigma) - \lambda_{\ell,k-\ell-2}(X-\sigma) \\ &= \binom{k+1}{\ell+1} B_{n,k,\ell} - \lambda_{\ell,k-\ell-2}(X-\sigma).\end{aligned}$$

Therefore

$$\lambda_{\ell,k-\ell-2}(X - \sigma) = 0. \tag{12}$$

Combining (11), (12) and the minimality of X, it follows that $\tilde{\beta}_k(X - \sigma) = 0$. Using (8) for the complex $X - \sigma$ it follows that

$$\begin{aligned}f_k(X-\sigma) &= \tilde{\beta}_k(X-\sigma) - \tilde{\beta}_{k-1}(X-\sigma) + \binom{n-1}{k} \\ &= \binom{n-1}{k} - B_{n,k,\ell} = F_{n,k,\ell}.\end{aligned}$$

Let $\tau \in \Delta_{n-1}(\ell)$. Using (8) for $\mathrm{lk}(X,\tau)$ we obtain

$$\tilde{\beta}_{k-\ell-1}(\mathrm{lk}(X-\sigma,\tau) - \tilde{\beta}_{k-\ell-2}(\mathrm{lk}(X-\sigma,\tau)) = f_{k-\ell-1}(\mathrm{lk}(X-\sigma,\tau)) - \binom{n-\ell-2}{k-\ell-1}. \tag{13}$$

Summing (13) over all $\tau \in \Delta_{n-1}(\ell)$ it follows that

$$\begin{aligned}\lambda_{\ell,k-\ell-1}(X-\sigma) &= \lambda_{\ell,k-\ell-1}(X-\sigma) - \lambda_{\ell,k-\ell-2}(X-\sigma) \\ &= \sum_{\tau\in\Delta_{n-1}(\ell)} f_{k-\ell-1}(\mathrm{lk}(X-\sigma,\tau)) - \binom{n}{\ell+1}\binom{n-\ell-2}{k-\ell-1} \\ &= \binom{k+1}{\ell+1} f_k(X-\sigma) - \binom{n}{\ell+1}\binom{n-\ell-2}{k-\ell-1} \\ &= \binom{k+1}{\ell+1} F_{n,k,\ell} - \binom{n}{\ell+1}\binom{n-\ell-2}{k-\ell-1} = 0.\end{aligned} \tag{14}$$

Choose a k-simplex $\sigma \neq \sigma' \in \mathrm{supp}(z)$ and an ℓ-simplex $\tau \in \sigma'(\ell) - \sigma(\ell)$. Then on one hand $\mathrm{lk}(X-\sigma,\tau) = \mathrm{lk}(X,\tau)$, hence by Claim 9

$$\tilde{\beta}_{k-\ell-1}(\mathrm{lk}(X-\sigma,\tau)) = \tilde{\beta}_{k-\ell-1}(\mathrm{lk}(X,\tau)) \neq 0.$$

On the other hand it follows from (14) that $\tilde{\beta}_{k-\ell-1}(\mathrm{lk}(X-\sigma,\tau)) = 0$, a contradiction. Thus $\tilde{\beta}_k(X) = 0$ and hence

$$f_k(X) = \tilde{\beta}_k(X) - \tilde{\beta}_{k-1}(X) + \binom{n-1}{k} = \binom{n-1}{k} - B_{n,k,\ell} = F_{n,k,\ell}.$$

(b) $\Rightarrow$ (c): Suppose $f_k(X) = F_{n,k,\ell}$. As $\lambda_{\ell,k-\ell-2}(X) = 0$, it follows that $\tilde{\beta}_{k-\ell-2}(\mathrm{lk}(X,\tau)) = 0$ and hence $f_{k-\ell-1}(\mathrm{lk}(X,\tau) \geq \binom{n-\ell-2}{k-\ell-1}$ for all $\tau \in \Delta_{n-1}(\ell)$. Therefore

$$\begin{aligned}F_{n,k,\ell} = f_k(X) &= \frac{1}{\binom{k+1}{\ell+1}} \sum_{\tau\in\Delta_{n-1}(\ell)} f_{k-\ell-1}(\mathrm{lk}(X,\tau)) \\ &\geq \frac{\binom{n}{\ell+1}\binom{n-\ell-2}{k-\ell-1}}{\binom{k+1}{\ell+1}} = F_{n,k,\ell}.\end{aligned} \tag{15}$$

Hence $f_{k-\ell-1}(\mathrm{lk}(X,\tau)) = \binom{n-\ell-2}{k-\ell-1}$ and therefore $\mathrm{lk}(X,\tau)$ is a $(k-\ell-1)$-hypertree for all $\tau \in \Delta_{n-1}(\ell)$.

(c) $\Rightarrow$ (a): Assume that $\mathrm{lk}(X,\tau)$ is a $(k-\ell-1)$-hypertree for all $\tau \in \Delta_{n-1}(\ell)$. Then, as in (15), it follows that $f_k(X) = F_{n,k,\ell}$. Furthermore, by (7)

$$\tilde{\beta}_k(X) \leq \frac{1}{\binom{k+1}{\ell+1}} \sum_{\tau\in X(\ell)} \tilde{\beta}_{k-\ell-1}(\mathrm{lk}(X,\tau)) = 0.$$

Hence

$$\tilde{\beta}_{k-1}(X) = \tilde{\beta}_k(X) - f_k(X) + \binom{n-1}{k} = \binom{n-1}{k} - F_{n,k,\ell} = B_{n,k,\ell}.$$

$\square$

4 Links of Sum Complexes

Let n be a prime and let A be a subset of the cyclic group $V=\mathbb{Z}_n$. Identify the vertex set of Δ_{n-1} with the elements of $\mathbb{Z}_n$. For $s \le n-2$ define the *sum complex* $Y_{A,s+1} \subset \Delta_{n-1}^{(s)}$ by

$$Y_{A,s+1} = \Delta_{n-1}^{(s-1)} \cup \{\sigma \subset \mathbb{Z}_n : |\sigma| = s+1 \text{ and } \sum_{x\in\sigma} x \in A\}.$$

The homology groups $\tilde{H}_*(Y_{A,s+1};\mathbb{K})$ were determined in [7, 8]. When A is a cyclic interval in $\mathbb{Z}_n$, the Betti numbers $\tilde{\beta}_*(Y_{A,s+1};\mathbb{K})$ do not depend on $\mathbb{K}$ and take the following simple form.

Theorem 10 ([7, 8])**.** *Let n be a prime and let $A=\{t,t+1,\ldots,t+r\}$ be an interval of size $r+1$ in $\mathbb{Z}_n$. Then for any field $\mathbb{K}$*

$$\tilde{\beta}_i(Y_{A,s+1};\mathbb{K}) = \begin{cases} \frac{s-r}{s+1}\binom{n-1}{s} & \text{if } i=s-1,\ r\le s, \\ \frac{r-s}{s+1}\binom{n-1}{s} & \text{if } i=s,\ r \ge s, \\ 0 & \text{otherwise.} \end{cases}$$

Let $\ell \le k-2$ and let $c_{k,\ell} = \frac{(\ell+1)(k-\ell)}{(k-\ell-1)!}$.

Proposition 11. *Let $B=\{0,\ldots,k-\ell-1\}$. Then for any $\tau \in \Delta_{n-1}(\ell)$*

$$\tilde{\beta}_{k-\ell-2}(\mathrm{lk}(Y_{B,k+1},\tau)) \le c_{k,\ell} n^{k-\ell-2}.$$

Proof: Let $y=\sum_{x\in\tau} x$ and let $C=\{b-y : b\in B\}$. Then

$$\mathrm{lk}(Y_{B,k+1},\tau) = Y_{C,k-\ell}[V-\tau] \subset Y_{C,k-\ell}.$$

Applying Theorem 10 with $A=C$, $r=s=k-\ell-1$ and $i\in\{s-1,s\}$, it follows that

$$\tilde{\beta}_{k-\ell-2}(Y_{C,k-\ell}) = \tilde{\beta}_{k-\ell-1}(Y_{C,k-\ell}) = 0.$$

Hence, the exact sequence

$$0=\tilde{H}_{k-\ell-1}(Y_{C,k-\ell}) \to \tilde{H}_{k-\ell-1}(Y_{C,k-\ell}, Y_{C,k-\ell}[V-\tau]) \to \tilde{H}_{k-\ell-2}(Y_{C,k-\ell}[V-\tau]) \to \tilde{H}_{k-\ell-2}(Y_{C,k-\ell}) = 0$$

implies

$$\tilde{H}_{k-\ell-1}(Y_{C,k-\ell}, Y_{C,k-\ell}[V-\tau]) \cong \tilde{H}_{k-\ell-2}(Y_{C,k-\ell}[V-\tau]). \tag{16}$$

For $a,c\in\mathbb{Z}_n$ let

$$\mathcal{F}_{a,c} = \{\sigma \in \Delta_{n-1}(k-\ell-1) : a\in\sigma\,,\ \sum_{x\in\sigma} x = c\}.$$

Note that any $\eta \in \Delta_{n-1}(k-\ell-2)$ is contained in at most one $\sigma\in\mathcal{F}_{a,c}$. Therefore

$$\begin{aligned} |\mathcal{F}_{a,c}| &= \frac{1}{k-\ell-1}|\{(\eta,\sigma) : a\in\eta\subset\sigma\in\mathcal{F}_{a,c}\,,\ |\eta|=k-\ell-1\}| \\ &\le \frac{1}{k-\ell-1}|\{\eta\in\Delta_{n-1}(k-\ell-2) : a\in\eta\}| \\ &= \frac{1}{k-\ell-1}\binom{n-1}{k-\ell-2} \le \frac{n^{k-\ell-2}}{(k-\ell-1)!}. \end{aligned} \tag{17}$$

Combining (16) and (17) it follows that

$$\begin{aligned}\tilde{\beta}_{k-\ell-2}(\mathrm{lk}(Y_{B,k+1},\tau)) &= \dim \tilde{H}_{k-\ell-2}(Y_{C,k-\ell}[V-\tau]) \\ &= \dim \tilde{H}_{k-\ell-1}(Y_{C,k-\ell}, Y_{C,k-\ell}[V-\tau]) \\ &\le f_{k-\ell-1}(Y_{C,k-\ell}) - f_{k-\ell-1}(Y_{C,k-\ell}[V-\tau]) \\ &= |\{\sigma \in \Delta_{n-1}(k-\ell-1) : \sigma \cap \tau \neq \emptyset\,,\, \sum_{x\in\sigma} x \in C\}| \\ &\le \sum_{(a,c)\in\tau\times C} |\mathcal{F}_{a,c}| \\ &\le \frac{(\ell+1)(k-\ell)}{(k-\ell-1)!} n^{k-\ell-2}.\end{aligned}$$

□

5 The Lower Bound

Proof of Theorem 5: For the case $\ell = k-1$ see the remark in Section 7. Assume that $\ell \le k-2$. Let n be a prime and let $B = \{0,\ldots,k-\ell-1\}$. For any $\tau \in \Delta_{n-1}(\ell)$ choose a set of $(k-\ell-1)$-dimensional simplices $S_\tau \subset \mathrm{lk}(\Delta_{n-1},\tau)(k-\ell-1)$ of size $|S_\tau| = \tilde{\beta}_{k-\ell-2}(\mathrm{lk}(Y_{B,k+1},\tau))$ such that

$$\tilde{\beta}_{k-\ell-2}(\mathrm{lk}(Y_{B,k+1},\tau) \cup S_\tau) = 0.$$

Proposition 11 implies that

$$|S_\tau| \le c_{k,\ell} n^{k-\ell-2}.$$

Let

$$X_{n,k,\ell} = Y_{B,k+1} \bigcup \{\tau \cup \eta : \tau \in \Delta_{n-1}(\ell), \eta \in S_\tau\}.$$

Then for all $\tau \in \Delta_{n-1}(\ell)$

$$\tilde{\beta}_{k-\ell-2}(\mathrm{lk}(X_{n,k,\ell},\tau)) = \tilde{\beta}_{k-\ell-2}(\mathrm{lk}(Y_{B,k+1},\tau) \cup S_\tau) = 0.$$

Applying Theorem 10 with $A = B$, $r = k-\ell-1$, $s = k$ and $i = s-1$, it follows that

$$\tilde{\beta}_{k-1}(Y_{B,k+1}) = \frac{\ell+1}{k+1}\binom{n-1}{k}.$$

Hence

$$\begin{aligned}\tilde{\beta}_{k-1}(X_{n,k,\ell}) &\ge \tilde{\beta}_{k-1}(Y_{B,k+1}) - \sum_{\tau\in\Delta_{n-1}(\ell)} |S_\tau| \\ &\ge \frac{\ell+1}{k+1}\binom{n-1}{k} - \binom{n}{\ell+1} c_{k,\ell} n^{k-\ell-2} \\ &\ge \frac{\binom{n-1}{\ell}\binom{n-\ell-2}{k-\ell}}{\binom{k+1}{\ell+1}}(1 - O(n^{-1})).\end{aligned}$$

□

6 Constructions of $J_{n,k}$ for $k \leq 3$

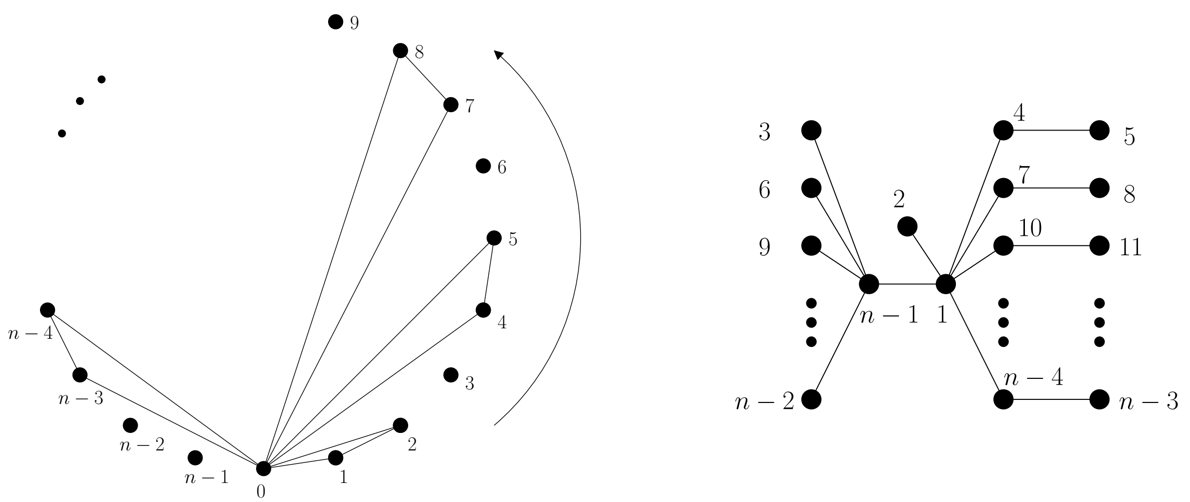
In this section we describe the constructions that establish Theorem 6.

(i) Let $k = 1$ and n be even. Let $J_{n,1}$ be a perfect matching on the vertex set $[n]$. Then $\tilde{\beta}_{-1}(\text{lk}(J_{n,1}, v)) = 0$ for all $v \in [n]$ and $\tilde{\beta}_0(J_{n,1}) = \frac{n}{2} - 1 = B_{n,1,0}$.

(ii) Let $k = 2$ and $n = 3t + 2$. Let $J_{n,2}$ be the 2-dimensional complex on the vertex set $\mathbb{Z}_n$ (see Figure 1(a)) given by

$$J_{n,2} = \Delta_{n-1}^{(1)} \cup \{\{i, i+3j+1, i+3j+2\} : i \in \mathbb{Z}_n \ , \ 0 \leq j \leq t-1\} .$$

Proposition 12. *$J_{n,2}$ satisfies $\tilde{\beta}_0(\text{lk}(J_{n,2}, v)) = 0$ for all $v \in \mathbb{Z}_n$ and $\tilde{\beta}_1(J_{n,2}) = \frac{1}{3}\binom{n-2}{2} = B_{n,2,0}$.*



(a) $J_{n,2}$ (b) $\text{lk}(J_{n,2}, 0)$

Figure 1

Proof: By Theorem 4 it suffices to show that for all $i \in \mathbb{Z}_n$ the graph $\text{lk}(J_{n,2}, i)$ is a tree on the vertex set $\mathbb{Z}_n - \{i\}$. By homogeneity it suffices to consider $\text{lk}(J_{n,2}, 0)$. It follows from the definition of $J_{n,2}$ that

$$\text{lk}(J_{n,2}, 0)(1) = A_0 \cup B_0 \cup C_0$$

where

$$A_0 = \{\{3j+1, 3j+2\} : 0 \leq j \leq t-1\} ,$$
$$B_0 = \{\{n-3j-1, 1\} : 0 \leq j \leq t-1\} = \{\{1, 3j+1\} : 1 \leq j \leq t\} ,$$
$$C_0 = \{\{n-3j-2, n-1\} : 0 \leq j \leq t-1\} = \{\{3j, n-1\} : 1 \leq j \leq t\} .$$

Thus, $\text{lk}(J_{n,2}, 0)$ is the tree on $\mathbb{Z}_n - \{0\}$ depicted in Figure 1(b).

□

(iii) Let $k = 3$ and $4 \leq n$ be even. Let $J_{n,3}$ be the 3-dimensional complex on the vertex set $\mathbb{Z}_n$ (see Figure 2) given by

$$J_{n,3} = \Delta_{n-1}^{(2)} \cup \left\{ \{i, i+\alpha, i+n/2, i+\frac{n}{2}+\alpha'\} : 0 \leq i < \frac{n}{2}\ ,\ 0 < \alpha, \alpha' < \frac{n}{2} \right\}.$$

Proposition 13. *$J_{n,3}$ satisfies $\tilde{\beta}_1(\mathrm{lk}(J_{n,3}, v)) = 0$ for all $v \in \mathbb{Z}_n$ and $\tilde{\beta}_2(J_{n,3}) = \frac{1}{4}\binom{n-2}{3} = B_{n,3,0}$.*

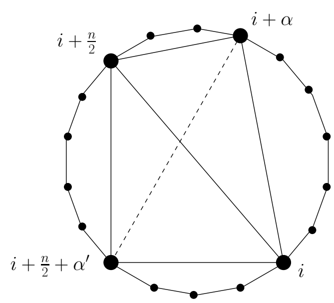


Figure 2: A 3-simplex in $J_{n,3}$

Proof: As in the proof of Proposition 12, it suffices to show that $\mathrm{lk}(J_{n,3}, 0)$ is a 2-hypertree on the vertex set $\mathbb{Z}_n - \{0\}$. We claim that $\mathrm{lk}(J_{n,3}, 0)$ is in fact collapsible. Partition the 2-simplices of $\mathrm{lk}(J_{n,3}, 0)$ into 3 disjoint families (see Figure 3):

$$\mathrm{lk}(J_{n,3}, 0)(2) = A_0 \cup B_0 \cup C_0,$$

where

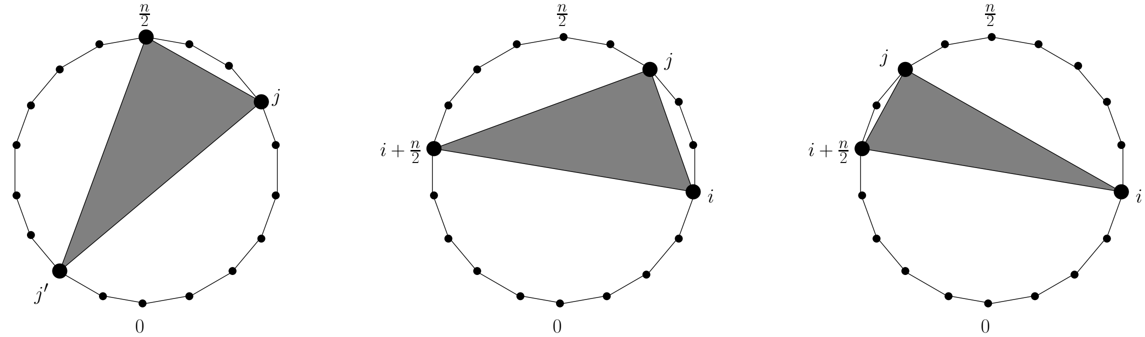
$$\begin{aligned} A_0 &= \left\{ \{j, n/2, j'\} : 0 < j < n/2 < j' < n \right\}, \\ B_0 &= \left\{ \{i, j, i+n/2\} : 0 < i < j < n/2 \right\}, \\ C_0 &= \left\{ \{i, j, i+n/2\} : 0 < i < n/2 < j < i+n/2 \right\}. \end{aligned}$$

If $0 < i < j < n/2$ then the edge $\{i, j\}$ is contained in the unique 2-simplex $\{i, j, i+n/2\} \in B_0$. If $0 < i < n/2 < j < i+n/2$ then the edge $\{j, i+n/2\}$ is contained in the unique 2-simplex $\{i, j, i+n/2\} \in C_0$. Collapsing all these edges and the corresponding 2-simplices, the resulting complex consists of all simplices in A_0 and their faces. This complex is a cone on the vertex $n/2$ and is therefore collapsible.

□

7 Concluding Remarks

We have shown that if $\Delta_{n-1}^{(k-1)} \subset X \subset \Delta_{n-1}^{(k)}$ satisfies $\tilde{\beta}_{k-\ell-2}(\mathrm{lk}(X, \tau); \mathbb{K}) = 0$ for all $\tau \in \Delta_{n-1}(\ell)$, then $\tilde{\beta}_{k-1}(X; \mathbb{K}) \leq B_{n,k,\ell}$. Furthermore, this bound is asymptotically tight for fixed k, ℓ and $n \to \infty$, and exact for $(k, \ell) = (1,0), (2,0), (3,0)$ and infinitely many n's. We suggest the following



(a) Type A_0 (b) Type B_0 (c) Type C_0

Figure 3: Three types of 2-simplices in $\mathrm{lk}(J_{n,3}, 0)$

Conjecture 14. *For any fixed $0 \le \ell < k$ there exists a constant $n_0(k,\ell)$ such that if $n \ge n_0(k,\ell)$ and if $B_{n,k,\ell}$ is an integer, then there exists a complex $\Delta_{n-1}^{(k-1)} \subset X \subset \Delta_{n-1}^{(k)}$ such that $\lambda_{\ell,k-\ell-2}(X) = 0$ and $\tilde{\beta}_{k-1}(X) = B_{n,k,\ell}$.*

Remarks:

1. Let $\ell = k-1$. It follows from Theorem 4 that $\Delta_{n-1}^{(k-1)} \subset X \subset \Delta_{n-1}^{(k)}$ satisfies both $\lambda_{\ell,k-\ell-2}(X) = \lambda_{k-1,-1}(X) = 0$ and $\tilde{\beta}_{k-1}(X) = B_{n,k,k-1} = (\frac{n}{k+1}-1)\binom{n-1}{k-1}$ iff each $(k-1)$-simplex $\tau \in \Delta_{n-1}(k-1)$ is contained in a unique k-simplex in $X(k)$, i.e. iff $S = X(k) \subset \binom{[n]}{k+1}$ is a Steiner system of type $S(k, k+1, n)$. Thus the case $\ell = k-1$ of Conjecture 14 follows from Keevash groundbreaking work [6] on the existence of designs.
2. It would be interesting and useful for various applications to interpolate between Corollary 2 and Garland's Theorem, in particular to obtain sharp upper bounds on the rational Betti number $\tilde{\beta}_{k-1}(X;\mathbb{Q})$ in terms of $\tilde{\mu}_{k-\ell-2}(X) = \min_{\tau \in X(\ell)} \mu_{k-\ell-2}(\mathrm{lk}(X,\tau))$ when $0 < \tilde{\mu}_{k-\ell-2}(X) \le \frac{\ell+1}{k}$.
3. The characterization given in Theorem 4 and the examples in Section 6 suggest some natural questions concerning hypertrees, e.g. for which $(k-1)$-hypertrees $\Delta_{n-2}^{(k-2)} \subset T \subset \Delta_{n-2}^{(k-1)}$ there exists a complex $\Delta_{n-1}^{(k-1)} \subset X \subset \Delta_{n-1}^{(k)}$ such that $\mathrm{lk}(X,v) \cong T$ for all vertices $v \in X(0)$?

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