Spectral Expansion of Random Sum Complexes

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Abstract

Let G be a finite abelian group of order n and let Δ_{n-1} denote the $(n-1)$-simplex on the vertex set G. The *sum complex* $X_{A,k}$ associated to a subset $A \subset G$ and $k < n$, is the k-dimensional simplicial complex obtained by taking the full $(k-1)$-skeleton of Δ_{n-1} together with all $(k+1)$-subsets $\sigma \subset G$ that satisfy $\sum_{x\in\sigma} x \in A$. Let $C^{k-1}(X_{A,k})$ denote the space of complex valued $(k-1)$-cochains of $X_{A,k}$. Let $L_{k-1} : C^{k-1}(X_{A,k}) \to C^{k-1}(X_{A,k})$ denote the reduced $(k-1)$-th Laplacian of $X_{A,k}$, and let $\mu_{k-1}(X_{A,k})$ be the minimal eigenvalue of L_{k-1}.

It is shown that if $k \geq 1$ and $\epsilon > 0$ are fixed, and A is a random subset of G of size $m = \lceil \frac{4k^2 \log n}{\epsilon^2} \rceil$, then

$$\Pr\left[\ \mu_{k-1}(X_{A,k}) < (1-\epsilon)m\ \right] = O\left(\frac{1}{n}\right).$$

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1 Introduction

The notion of expansion in graphs plays a key role in a variety of questions in both pure and applied mathematics, with numerous applications ranging from randomization reduction in probabilistic algorithms to the construction of good error correcting codes (see e.g. [5, 7]). In view of the ubiquity of expander graphs, there is a growing interest in understanding different notions of expansion for higher dimensional simplicial complexes (see e.g. Lubotzky's ICM lecture [8]). In this paper we study the generic spectral expansion of certain arithmetically constructed simplicial complexes called sum complexes.

Let G be a finite abelian group of order n and let Δ_{n-1} denote the $(n-1)$-simplex on the vertex set G. The *sum complex* $X_{A,k}$ associated to a subset $A \subset G$ and $k < n$, is the k-dimensional simplicial complex obtained by taking the full $(k-1)$-skeleton of Δ_{n-1}, together with all $(k+1)$-subsets $\sigma \subset G$ that satisfy $\sum_{x\in\sigma} x \in A$.

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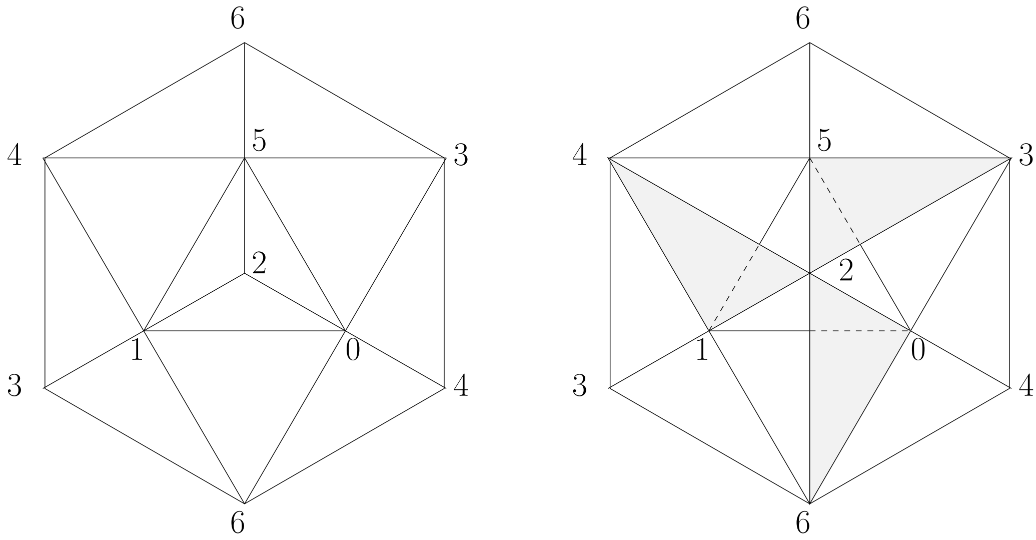

(a) A 7-point triangulation of $\mathbb{RP}^2$

(b) $X_{A,2}$ for $A = \{0, 1, 3\} \subset \mathbb{Z}_7$

Figure 1

Example: Let $G = \mathbb{Z}_7$ be the cyclic group of order 7, and let $A = \{0, 1, 3\}$. The sum complex $X_{A,2}$ is depicted in Figure 1b). Note that $X_{A,2}$ is obtained from a 7-point triangulation of the real projective plane $\mathbb{RP}^2$ (Figure 1a) by adding the faces $\{2, 3, 5\}$, $\{0, 2, 6\}$ and $\{1, 2, 4\}$. $X_{A,2}$ is clearly homotopy equivalent to $\mathbb{RP}^2$.

The sum complexes $X_{A,k}$ may be viewed as k-dimensional analogues of Cayley graphs over G. They were defined and studied (for cyclic groups) in [6, 9], where some of their combinatorial and topological properties were established. For example, for $G = \mathbb{Z}_p$, the cyclic group of prime order p, the homology of $X_{A,k}$ was determined in [6] for coefficient fields $\mathbb{F}$ of characteristic coprime to p, and in [9] for general $\mathbb{F}$. In particular, for $\mathbb{F} = \mathbb{C}$ we have the following

Theorem 1.1 ([6, 9])**.** *Let $p > 2$ be a prime and let $A \subset \mathbb{Z}_p$ such that $|A| = m$. Then for $1 \le k < p - 1$*

$$\dim \tilde{H}_{k-1}(X_{A,k}; \mathbb{C}) = \begin{cases} 0 & \text{if } m \ge k+1, \\ (1 - \frac{m}{k+1})\binom{p-1}{k} & \text{if } m \le k+1. \end{cases}$$

For a simplicial complex X and $k \ge -1$ let $C^k(X)$ denote the space of complex valued simplicial k-cochains of X and let $d_k : C^k(X) \to C^{k+1}(X)$ denote the coboundary operator. For $k \ge 0$ define the reduced k-th Laplacian of X by $L_k(X) = d_{k-1}d_{k-1}^* + d_k^* d_k$ (see section 2 for details). The minimal eigenvalue of $L_k(X)$, denoted by $\mu_k(X)$, is the *k-th spectral gap* of X.

Theorem 1.1 implies that if A is a subset of $G = \mathbb{Z}_p$ of size $|A| = m \ge k + 1$, then $\tilde{H}_{k-1}(X_{A,k}; \mathbb{C}) = 0$ and hence $\mu_{k-1}(X_{A,k}) > 0$. Returning to the case of general finite abelian groups G, it is then natural to ask for better lower bounds on the spectral gap $\mu_{k-1}(X_{A,k})$. Note that any $(k-1)$-simplex $\sigma \in \Delta_{n-1}$ is contained in at most m simplices of $X_{A,k}$ of dimension k, and therefore $\mu_{k-1}(X_{A,k}) \le m+k$ (see (2) in Section 2). Let log denote natural logarithm. Our main result asserts, roughly speaking, that if $k \geq 1$ and $\epsilon > 0$ are fixed and A is a random subset of G of size $m = \lceil c(k,\epsilon)\log n \rceil$, then $\mu_{k-1}(X_{A,k}) > (1-\epsilon)m$ asymptotically almost surely (a.a.s.). The precise statement is as follows.

Theorem 1.2. *Let k and $\epsilon > 0$ be fixed. Let G be an abelian group of order $n > \frac{2^{10}k^8}{\epsilon^8}$, and let A be a random subset of G of size $m = \lceil \frac{4k^2 \log n}{\epsilon^2} \rceil$. Then*

$$\Pr\left[\ \mu_{k-1}(X_{A,k}) < (1-\epsilon)m\ \right] < \frac{6}{n}.$$

Remarks:
1. Alon and Roichman [2] proved that for any $\epsilon > 0$ there exists a constant $c(\epsilon) > 0$ such that for any group G of order n, if S is a random subset of G of size $\lceil c(\epsilon)\log n \rceil$ and $m = |S \cup S^{-1}|$, then the spectral gap of the m-regular Cayley graph $\mathrm{C}(G, S \cup S^{-1})$ is a.a.s. at least $(1-\epsilon)m$. Theorem 1.2 may be viewed as a sort of high dimensional analogue of the Alon-Roichman theorem for abelian groups.

2. For $0 \leq q \leq 1$ let $Y_k(n,q)$ denote the probability space of random complexes obtained by taking the full $(k-1)$-skeleton of Δ_{n-1} and then adding each k-simplex independently with probability q. Let $d = q(n-k)$ denote the expected number of k-simplices containing a fixed $(k-1)$-simplex. Gundert and Wagner [4] proved that for any $\delta > 0$ there exists a $C = C(\delta)$ such that if $q \geq (k+\delta)\log n/n$, then $Y \in Y_k(n,q)$ satisfies a.a.s. $\mu_{k-1}(Y) \geq d - C\sqrt{d}$.

3. The assumption on m in Theorem 1.2 cannot in general be replaced by $m = o(\log n)$, see Proposition 4.2.

The paper is organized as follows. In Section 2 we recall some basic properties of high dimensional Laplacians and their eigenvalues. In Section 3 we study the Fourier images of $(k-1)$-cocycles of sum complexes, and obtain a lower bound (Theorem 3.1) on $\mu_{k-1}(X_{A,k})$, in terms of the Fourier transform of the indicator function of A. This bound is the key ingredient in the proof of Theorem 1.2 given in Section 4. We conclude in Section 5 with some remarks and open problems.

2 Laplacians and their Eigenvalues

Let X be a finite simplicial complex on the vertex set V. Let $X^{(k)} = \{\sigma \in X : \dim \sigma \leq k\}$ be the k-th skeleton of X, and let $X(k)$ denote the set of k-dimensional simplices in X, each taken with an arbitrary but fixed orientation. The face numbers of X are $f_k(X) = |X(k)|$. A simplicial k-cochain is a complex valued skew-symmetric function on all ordered k-simplices of X. For $k \geq 0$ let $C^k(X)$ denote the space of k-cochains on X. The i-face of an ordered $(k+1)$-simplex $\sigma = [v_0, \ldots, v_{k+1}]$ is the ordered k-simplex $\sigma_i = [v_0, \ldots, \widehat{v_i}, \ldots, v_{k+1}]$. The coboundary operator $d_k : C^k(X) \to C^{k+1}(X)$ is given by

$$d_k\phi(\sigma) = \sum_{i=0}^{k+1} (-1)^i \phi(\sigma_i) \quad .$$

It will be convenient to augment the cochain complex $\{C^i(X)\}_{i=0}^{\infty}$ with the (-1)-degree term $C^{-1}(X) = \mathbb{C}$ with the coboundary map $d_{-1} : C^{-1}(X) \to C^0(X)$ given by $d_{-1}(a)(v) = a$

for $a \in \mathbb{C}$, $v \in V$. Let $Z^k(X) = \ker(d_k)$ denote the space of k-cocycles and let $B^k(X) = \mathrm{Im}(d_{k-1})$ denote the space of k-coboundaries. For $k \geq 0$ let $\tilde{H}^k(X) = Z^k(X)/B^k(X)$ denote the k-th reduced cohomology group of X with complex coefficients. For each $k \geq -1$ endow $C^k(X)$ with the standard inner product $(\phi, \psi)_X = \sum_{\sigma \in X(k)} \phi(\sigma)\overline{\psi(\sigma)}$ and the corresponding L^2 norm $||\phi||_X = (\phi, \phi)^{1/2}$.
Let $d_k^* : C^{k+1}(X) \to C^k(X)$ denote the adjoint of d_k with respect to these standard inner products. The reduced k-th Laplacian of X is the mapping

$$L_k(X) = d_{k-1}d_{k-1}^* + d_k^* d_k : C^k(X) \to C^k(X).$$

The k-th Laplacian $L_k(X)$ is a positive semi-definite Hermitian operator on $C^k(X)$. Its minimal eigenvalue, denoted by $\mu_k(X)$, is the *k-th spectral gap* of X. For two ordered simplices $\alpha \subset \beta$ let $(\beta : \alpha) \in \{\pm 1\}$ denote the incidence number between β and α. Let $\deg(\beta)$ denote the number of simplices γ of dimensional $\dim \beta + 1$ that contain β . For an ordered k-simplex $\sigma = [v_0, \ldots, v_k] \in X(k)$, let $1_\sigma \in C^k(X)$ be the indicator k-cochain of σ, i.e. $1_\sigma(u_0, \ldots, u_k) = \mathrm{sign}(\pi)$ if $u_i = v_{\pi(i)}$ for some permutation $\pi \in S_{k+1}$, and zero otherwise. By a simple computation (see e.g. (3.4) in [10]), the matrix representation of L_k with respect to the standard basis $\{1_\sigma\}_{\sigma \in X(k)}$ of $C^k(X)$ is given by

$$L_k(X)(\sigma, \tau) = \begin{cases} \deg(\sigma) + k + 1 & \sigma = \tau, \\ (\sigma : \sigma \cap \tau) \cdot (\tau : \sigma \cap \tau) & |\sigma \cap \tau| = k \ , \ \sigma \cup \tau \notin X, \\ 0 & \text{otherwise.} \end{cases} \tag{1}$$

Remarks:
1. By (1)

$$\mathrm{tr}\, L_k(X) = \sum_{\sigma \in X(k)} (\deg(\sigma) + k + 1) = (k+2) f_{k+1}(X) + (k+1) f_k(X).$$

Hence

$$\begin{aligned} \mu_k(X) &\leq \frac{\mathrm{tr}\, L_k(X)}{f_k(X)} \leq (k+2)\frac{f_{k+1}(X)}{f_k(X)} + k + 1 \\ &\leq \max_{\sigma \in X(k)} \deg(\sigma) + k + 1. \end{aligned} \tag{2}$$

2. The matrix representation of $L_0(X)$ is equal to $J + L$, where J is the $V \times V$ all ones matrix, and L is the graph Laplacian of the 1-skeleton $X^{(1)}$ of X. In particular, $\mu_0(X)$ is equal to the graphical spectral gap $\lambda_2(X^{(1)})$.

In the rest of this section we record some well known properties of the coboundary operators and Laplacians on the $(n-1)$-simplex Δ_{n-1} (Claim 2.1), and on subcomplexes of Δ_{n-1} that contain its full $(k-1)$-skeleton (Proposition 2.2). Let I denote the identity operator on $C^{k-1}(\Delta_{n-1})$.

Claim 2.1.
(i) The $(k-1)$-Laplacian on Δ_{n-1} satisfies $L_{k-1}(\Delta_{n-1}) = n \cdot \mathrm{I}$.
(ii) There is an orthogonal decomposition

$$C^{k-1}(\Delta_{n-1}) = \ker d_{k-2}^* \oplus \mathrm{Im}\, d_{k-2}.$$

(iii) The operators $P = \mathrm{I} - \frac{1}{n} d_{k-2} d_{k-2}^*$ and $Q = \frac{1}{n} d_{k-2} d_{k-2}^*$ are, respectively, the orthogonal projections of $C^{k-1}(\Delta_{n-1})$ onto $\ker d_{k-2}^*$ and onto $\mathrm{Im}\, d_{k-2}$.

Proof. Part (i) follows from (1). Next observe that $\ker d_{k-2}^* \perp \mathrm{Im}\, d_{k-2}$ and

$$\dim \ker d_{k-2}^* + \dim \mathrm{Im}\, d_{k-2} = \dim \ker d_{k-2}^* + \dim \mathrm{Im}\, d_{k-2}^* = \dim C^{k-1}(\Delta_{n-1}).$$

This implies (ii).
(iii) By (i):

$$n \cdot \mathrm{I} = L_{k-1}(\Delta_{n-1}) = d_{k-2} d_{k-2}^* + d_{k-1}^* d_{k-1},$$

and hence

$$n d_{k-2}^* = d_{k-2}^* d_{k-2} d_{k-2}^* + d_{k-2}^* d_{k-1}^* d_{k-1} = d_{k-2}^* d_{k-2} d_{k-2}^*.$$

It follows that

$$d_{k-2}^* P = d_{k-2}^* - \frac{1}{n} d_{k-2}^* d_{k-2} d_{k-2}^* = 0,$$

and therefore $\mathrm{Im}\, P \subset \ker d_{k-2}^*$. Since clearly $\mathrm{Im}\, Q \subset \mathrm{Im}\, d_{k-2}$, it follows that P is the projection onto $\ker_{k-2}^*$ and Q is the projection onto $\mathrm{Im}\, d_{k-2}$.

$\square$

The variational characterization of the eigenvalues of Hermitian operators implies that for any complex X

$$\begin{aligned} \mu_{k-1}(X) &= \min \left\{ \frac{(L_{k-1}\phi, \phi)_X}{(\phi, \phi)_X} : 0 \neq \phi \in C^{k-1}(X) \right\} \\ &= \min \left\{ \frac{\|d_{k-2}^* \phi\|_X^2 + \|d_{k-1}\phi\|_X^2}{\|\phi\|_X^2} : 0 \neq \phi \in C^{k-1}(X) \right\}. \end{aligned} \tag{3}$$

When X contains the full $(k-1)$-skeleton we have the following stronger statement.

Proposition 2.2. *Let $\Delta_{n-1}^{(k-1)} \subset X \subset \Delta_{n-1}$. Then*

$$\mu_{k-1}(X) = \min \left\{ \frac{\|d_{k-1}\phi\|_X^2}{\|\phi\|_X^2} : 0 \neq \phi \in \ker d_{k-2}^* \right\}. \tag{4}$$

Proof. The $\leq$ statement in (4) follows directly from (3). We thus have to show the reverse inequality. First note that if $\psi \in C^{k-1}(X)$ then by Claim 2.1(i)

$$\begin{aligned} \|d_{k-1}\psi\|_X^2 &\leq \|d_{k-1}\psi\|_{\Delta_{n-1}}^2 \\ &\leq \|d_{k-2}^*\psi\|_{\Delta_{n-1}}^2 + \|d_{k-1}\psi\|_{\Delta_{n-1}}^2 \\ &= (L_{k-1}(\Delta_{n-1})\psi, \psi)_{\Delta_{n-1}} = n\|\psi\|_X^2. \end{aligned} \tag{5}$$

Furthermore, if $\phi \in C^{k-1}(X)$ then by Claim 2.1(iii)

$$d_{k-1}\phi = d_{k-1}P\phi + d_{k-1}Q\phi = d_{k-1}P\phi, \tag{6}$$

and

$$\begin{aligned}\|d_{k-2}^*\phi\|_X^2 &= (d_{k-2}^*\phi, d_{k-2}^*\phi)_X = (\phi, d_{k-2}d_{k-2}^*\phi)_X \\ &= n(\phi, Q\phi)_X = n\|Q\phi\|_X^2.\end{aligned} \tag{7}$$

It follows that

$$\begin{aligned}\mu_{k-1}(X) &= \min\left\{\frac{\|d_{k-2}^*\phi\|_X^2 + \|d_{k-1}\phi\|_X^2}{\|\phi\|_X^2} : 0 \neq \phi \in C^{k-1}(X)\right\} \\ &= \min\left\{\frac{n\|Q\phi\|_X^2 + \|d_{k-1}(P\phi)\|_X^2}{\|Q\phi\|_X^2 + \|P\phi\|_X^2} : 0 \neq \phi \in C^{k-1}(X)\right\} \\ &\geq \min\left\{\frac{\|d_{k-1}(P\phi)\|_X^2}{\|P\phi\|_X^2} : 0 \neq \phi \in C^{k-1}(X)\right\} \\ &= \min\left\{\frac{\|d_{k-1}\psi\|_X^2}{\|\psi\|_X^2} : 0 \neq \psi \in \ker d_{k-2}^*\right\},\end{aligned}$$

where the first equality is (3), the second equality follows from (6) and (7), the third inequality follows from (5) with $\psi = P\phi$, and the last equality is a consequence of Claim 2.1(iii).

□

3 Fourier Transform and Spectral Gaps

Let G be a finite abelian group of order n. Let $\mathcal{L}(G)$ denote the space of complex valued functions on G with the standard inner product $(\phi, \psi) = \sum_{x \in G} \phi(x)\overline{\psi(x)}$ and the corresponding L^2 norm $\|\phi\| = (\phi, \phi)^{1/2}$. Let $\widehat{G}$ be the character group of G. The *Fourier Transform* of $\phi \in \mathcal{L}(G)$, is the function $\widehat{\phi} \in \mathcal{L}(\widehat{G})$ whose value on the character $\chi \in \widehat{G}$ is given by $\widehat{\phi}(\chi) = \sum_{x \in G} \phi(x)\chi(-x)$. For $\phi, \psi \in \mathcal{L}(G)$ we have the Parseval identity $(\widehat{\phi}, \widehat{\psi}) = n(\phi, \psi)$, and in particular $\|\widehat{\phi}\|^2 = n\|\phi\|^2$.

Let G^k denote the direct product $G \times \cdots \times G$ (k times). The character group $\widehat{G^k}$ is naturally identified with $\widehat{G}^k$. Let $\widetilde{\mathcal{L}}(G^k)$ denote the subspace of skew-symmetric functions in $\mathcal{L}(G^k)$. Then $\widetilde{\mathcal{L}}(G^k)$ is mapped by the Fourier transform onto $\widetilde{\mathcal{L}}(\widehat{G}^k)$. Recall that Δ_{n-1} is the simplex on the vertex set G, and let $X \subset \Delta_{n-1}$ be a simplicial complex that contains the full $(k-1)$-skeleton of Δ_{n-1}. As sets, we will identify $C^{k-1}(X) = C^{k-1}(\Delta_{n-1})$ with $\widetilde{\mathcal{L}}(G^k)$. Note, however, that the inner products and norms defined on $C^{k-1}(\Delta_{n-1})$ and on $\widetilde{\mathcal{L}}(G^k)$ differ by multiplicative constants: If $\phi, \psi \in C^{k-1}(\Delta_{n-1}) = \widetilde{\mathcal{L}}(G^k)$ then $(\phi, \psi) = k!(\phi, \psi)_{\Delta_{n-1}}$ and $\|\phi\| = \sqrt{k!}\|\phi\|_{\Delta_{n-1}}$.

Let $A \subset G$ and let $k < n = |G|$. Let $\chi_0 \in \widehat{G}$ denote the trivial character of G and let $\widehat{G}_+ = \widehat{G} \setminus \{\chi_0\}$. Let $1_A \in \mathcal{L}(G)$ denote the indicator function of A, i.e. $1_A(x) = 1$ if $x \in A$ and zero otherwise. Then $\widehat{1_A}(\eta) = \sum_{a \in A} \eta(-a)$ for $\eta \in \widehat{G}$. The main result of this section is the following lower bound on the spectral gap of $X_{A,k}$.

Theorem 3.1.

$$\mu_{k-1}(X_{A,k}) \geq |A| - k \max\{|\widehat{1_A}(\eta)| : \eta \in \widehat{G}_+\}.$$

The proof of Theorem 3.1 will be based on two preliminary results, Propositions 3.2 and 3.4. The first of these is the following Fourier theoretic characterization of $\ker d_{k-2}^* \subset C^{k-1}(\Delta_{n-1})$.

Proposition 3.2. *Let $\phi \in C^{k-1}(\Delta_{n-1}) = \widetilde{\mathcal{L}}(G^k)$. Then $d_{k-2}^*\phi = 0$ iff $\mathrm{supp}(\widehat{\phi}) \subset (\widehat{G}_+)^k$.*

Proof. If $d_{k-2}^*\phi = 0$ then for all $(x_1, \dots, x_{k-1}) \in G^{k-1}$:

$$0 = d_{k-2}^*\phi(x_1, \dots, x_{k-1}) = \sum_{x_0 \in G} \phi(x_0, x_1, \dots, x_{k-1}).$$

Let $(\chi_1, \dots, \chi_{k-1})$ be an arbitrary element of $\widehat{G}^{k-1}$ and write $\chi = (\chi_0, \chi_1, \dots, \chi_{k-1}) \in \widehat{G}^k$. Then

$$\begin{aligned}
\widehat{\phi}(\chi) &= \sum_{(x_0,\dots,x_{k-1}) \in G^k} \phi(x_0, x_1, \dots, x_{k-1}) \prod_{j=1}^{k-1} \chi_j(-x_j) \\
&= \sum_{(x_1,\dots,x_{k-1}) \in G^{k-1}} \left(\sum_{x_0 \in G} \phi(x_0, x_1, \dots, x_{k-1}) \right) \prod_{j=1}^{k-1} \chi_j(-x_j) = 0.
\end{aligned}$$

The skew-symmetry of $\widehat{\phi}$ thus implies that $\mathrm{supp}(\widehat{\phi}) \subset (\widehat{G}_+)^k$. The other direction is similar.

$\square$

For the rest of this section let $X = X_{A,k}$. Fix $\phi \in C^{k-1}(X) = \widetilde{\mathcal{L}}(G^k)$. Our next step is to obtain a lower bound on $\|d_{k-1}\phi\|_X$ via the Fourier transform $\widehat{d_{k-1}\phi}$. For $a \in G$ define a function $f_a \in \widetilde{\mathcal{L}}(G^k)$ by

$$\begin{aligned}
f_a(x_1, \dots, x_k) &= d_{k-1}\phi\big(a - \sum_{i=1}^{k} x_i, x_1, \dots, x_k\big) \\
&= \phi(x_1, \dots, x_k) + \sum_{i=1}^{k} (-1)^i \phi\big(a - \sum_{j=1}^{k} x_j, x_1, \dots, \hat{x}_i, \dots, x_k\big).
\end{aligned}$$

By the Parseval identity

$$\begin{aligned}
\|d_{k-1}\phi\|_X^2 &= \sum_{\tau\in X(k)} |d_{k-1}\phi(\tau)|^2 \\
&= \frac{1}{(k+1)!} \sum_{\{(x_0,\ldots,x_k)\in G^{k+1}:\{x_0,\ldots,x_k\}\in X\}} |d_{k-1}\phi(x_0,\ldots,x_k)|^2 \\
&= \frac{1}{(k+1)!} \sum_{a\in A} \sum_{x=(x_1,\ldots,x_k)\in G^k} |d_{k-1}\phi(a-\sum_{i=1}^k x_i, x_1,\ldots,x_k)|^2 \qquad (8)\\
&= \frac{1}{(k+1)!} \sum_{a\in A}\sum_{x\in G^k} |f_a(x)|^2 \\
&= \frac{1}{n^k(k+1)!} \sum_{a\in A}\sum_{\chi\in\widehat{G}^k} |\widehat{f_a}(\chi)|^2.
\end{aligned}$$

We next find an expression for $\widehat{f_a}(\chi)$. Let T be the automorphism of $\widehat{G}^k$ given by

$$T(\chi_1,\ldots,\chi_k) = (\chi_2\chi_1^{-1},\ldots,\chi_k\chi_1^{-1},\chi_1^{-1}) \ .$$

Then $T^{k+1}=I$ and for $1\le i\le k$

$$T^i(\chi_1,\ldots,\chi_k) = (\chi_{i+1}\chi_i^{-1},\ldots,\chi_k\chi_i^{-1},\chi_i^{-1},\chi_1\chi_i^{-1},\ldots,\chi_{i-1}\chi_i^{-1}). \qquad (9)$$

The following result is a slight extension of Claim 2.2 in [6]. Recall that χ_0 is the trivial character of G.

Claim 3.3. *Let $\chi=(\chi_1,\ldots,\chi_k)\in\widehat{G}^k$. Then*

$$\widehat{f_a}(\chi) = \sum_{i=0}^k (-1)^{ki}\chi_i(-a)\widehat{\phi}(T^i\chi). \qquad (10)$$

Proof. For $1\le i\le k$ let $\psi_i\in\mathcal{L}(G^k)$ be given by

$$\psi_i(x_1,\ldots,x_k) = \phi\big(a-\sum_{j=1}^k x_j, x_1,\ldots,\hat{x}_i,\ldots,x_k\big).$$

Then

$$\widehat{\psi_i}(\chi) = \sum_{(x_1,\ldots,x_k)\in G^k} \phi\big(a-\sum_{j=1}^k x_j, x_1,\ldots,\hat{x}_i,\ldots,x_k\big)\prod_{j=1}^k \chi_j(-x_j) \ .$$

Substituting

$$y_j = \begin{cases} a-\sum_{\ell=1}^k x_\ell & j=1, \\ x_{j-1} & 2\le j\le i, \\ x_j & i+1\le j\le k, \end{cases}$$

it follows that

$$\prod_{j=1}^k \chi_j(-x_j) = \chi_i^{-1}(a-y_1)\prod_{j=2}^i(\chi_i^{-1}\chi_{j-1})(-y_j)\prod_{j=i+1}^k(\chi_i^{-1}\chi_j)(-y_j) \ .$$

Therefore

$$\begin{aligned}\widehat{\psi}_i(\chi) &= \chi_i(-a) \sum_{y=(y_1,\ldots,y_k)\in G^k} \phi(y)\chi_i^{-1}(-y_1)\prod_{j=2}^{i}(\chi_{j-1}\chi_i^{-1})(-y_j)\prod_{j=i+1}^{k}(\chi_j\chi_i^{-1})(-y_j) \\ &= \chi_i(-a)\widehat{\phi}(\chi_i^{-1},\chi_1\chi_i^{-1},\ldots,\chi_{i-1}\chi_i^{-1},\chi_{i+1}\chi_i^{-1},\ldots,\chi_k\chi_i^{-1}) \\ &= \chi_i(-a)(-1)^{i(k-i)}\widehat{\phi}(T^i\chi)\ .\end{aligned} \tag{11}$$

Now (10) follows from (11) since $f_a = \phi + \sum_{i=1}^{k}(-1)^i\psi_i$.

□

For $\phi \in \widetilde{\mathcal{L}}(G^k)$ and $\chi = (\chi_1,\ldots,\chi_k) \in \widehat{G}^k$ let

$$\begin{aligned}D(\phi,\chi) &= \{\chi_i\chi_j^{-1} : 0 \le i < j \le k\ ,\ \widehat{\phi}(T^i\chi)\widehat{\phi}(T^j\chi) \neq 0\} \\ &= \{\chi_j^{-1} : 1 \le j \le k\ ,\ \widehat{\phi}(\chi)\widehat{\phi}(T^j\chi) \neq 0\} \cup \{\chi_i\chi_j^{-1} : 1 \le i < j \le k\ ,\ \widehat{\phi}(T^i\chi)\widehat{\phi}(T^j\chi) \neq 0\}.\end{aligned}$$

Let $D(\phi) = \bigcup_{\chi\in\widehat{G}^k} D(\phi,\chi) \subset \widehat{G}$. The main ingredient in the proof of Theorem 3.1 is the following

Proposition 3.4.

$$\|d_{k-1}\phi\|_X^2 \ge \left(|A| - k\max_{\eta\in D(\phi)}|\widehat{1_A}(\eta)|\right)\|\phi\|_X^2.$$

Proof. Let $\chi = (\chi_1,\ldots,\chi_k) \in \widehat{G}^k$. By Claim 3.3

$$\begin{aligned}\sum_{a\in A}|\widehat{f_a}(\chi)|^2 &= \sum_{a\in A}|\sum_{i=0}^{k}(-1)^{ki}\chi_i(-a)\widehat{\phi}(T^i\chi)|^2 \\ &= \sum_{a\in A}\sum_{i,j=0}^{k}(-1)^{k(i+j)}(\chi_i\chi_j^{-1})(-a)\widehat{\phi}(T^i\chi)\overline{\widehat{\phi}(T^j\chi)} \\ &= |A|\sum_{i=0}^{k}|\widehat{\phi}(T^i\chi)|^2 + 2\,\mathrm{Re}\sum_{a\in A}\sum_{0\le i<j\le k}(-1)^{k(i+j)}(\chi_i\chi_j^{-1})(-a)\widehat{\phi}(T^i\chi)\overline{\widehat{\phi}(T^j\chi)} \\ &= |A|\sum_{i=0}^{k}|\widehat{\phi}(T^i\chi)|^2 + 2\,\mathrm{Re}\sum_{0\le i<j\le k}(-1)^{k(i+j)}\widehat{1_A}(\chi_i\chi_j^{-1})\widehat{\phi}(T^i\chi)\overline{\widehat{\phi}(T^j\chi)} \\ &\ge |A|\sum_{i=0}^{k}|\widehat{\phi}(T^i\chi)|^2 - 2\max_{\eta\in D(\phi,\chi)}|\widehat{1_A}(\eta)|\sum_{0\le i<j\le k}|\widehat{\phi}(T^i\chi)|\cdot|\widehat{\phi}(T^j\chi)|.\end{aligned} \tag{12}$$

Using (8) and summing (12) over all $\chi \in \widehat{G}^k$ it follows that

$$
\begin{aligned}
& n^k(k+1)!\|d_{k-1}\phi\|_X^2 = \sum_{a\in A}\sum_{\chi\in\widehat{G}^k} |\widehat{f}_a(\chi)|^2 \\
& \geq |A|\sum_{i=0}^{k}\sum_{\chi\in\widehat{G}^k} |\widehat{\phi}(T^i\chi)|^2 - 2\max_{\eta\in D(\phi)} |\widehat{1_A}(\eta)| \sum_{0\leq i<j\leq k}\sum_{\chi\in\widehat{G}^k} |\widehat{\phi}(T^i\chi)|\cdot|\widehat{\phi}(T^j\chi)| \\
& \geq (k+1)|A|\sum_{\chi\in\widehat{G}^k} |\widehat{\phi}(\chi)|^2 - k(k+1)\max_{\eta\in D(\phi)} |\widehat{1_A}(\eta)| \sum_{\chi\in\widehat{G}^k} |\widehat{\phi}(\chi)|^2 \\
& = (k+1)n^k \sum_{x\in G^k} |\phi(x)|^2 \left(|A| - k\max_{\eta\in D(\phi)} |\widehat{1_A}(\eta)|\right) \\
& = (k+1)!n^k\|\phi\|_X^2 \left(|A| - k\max_{\eta\in D(\phi)} |\widehat{1_A}(\eta)|\right).
\end{aligned}
$$

□

Proof of Theorem 3.1. Let $0 \neq \phi \in C^{k-1}(X_{A,k}) = \widetilde{\mathcal{L}}(G^k)$ such that $d_{k-2}^*\phi = 0$. Proposition 3.2 implies that

$$\operatorname{supp}(\widehat{\phi}) \subset (\widehat{G}_+)^k. \tag{13}$$

We claim that $\chi_0 \notin D(\phi)$. Suppose to the contrary that $\chi_0 \in D(\phi)$. Then there exists a $\chi = (\chi_1, \ldots, \chi_k) \in \widehat{G}^k$ such that $\chi_0 \in D(\phi, \chi)$, i.e. $\chi_i = \chi_j$ for some $0 \leq i < j \leq k$ such that $\widehat{\phi}(T^i\chi)\widehat{\phi}(T^j\chi) \neq 0$. We consider two cases:

- If $i = 0$ then $\chi_j = \chi_i = \chi_0$ and therefore

  $$0 \neq \widehat{\phi}(T^i\chi) = \widehat{\phi}(\chi) = \widehat{\phi}(\chi_1, \ldots, \chi_{j-1}, \chi_0, \chi_{j+1}, \ldots, \chi_k),$$

  in contradiction of (13).

- If $i \geq 1$ then $\chi_j\chi_i^{-1} = \chi_0$, and by (9)

  $$
  \begin{aligned}
  0 \neq \widehat{\phi}(T^i\chi) &= \widehat{\phi}(\chi_{i+1}\chi_i^{-1}, \ldots, \chi_k\chi_i^{-1}, \chi_i^{-1}, \chi_1\chi_i^{-1}, \ldots, \chi_{i-1}\chi_i^{-1}) \\
  &\widehat{\phi}(\chi_{i+1}\chi_i^{-1}, \ldots, \chi_{j-1}\chi_i^{-1}, \chi_0, \chi_{j+1}\chi_i^{-1}, \ldots, \chi_k\chi_i^{-1}, \chi_i^{-1}, \chi_1\chi_i^{-1}, \ldots, \chi_{i-1}\chi_i^{-1}),
  \end{aligned}
  $$

  again in contradiction of (13).

We have thus shown that $D(\phi) \subset \widehat{G}_+$. Combining Propositions 2.2 and 3.4 we obtain

$$
\begin{aligned}
\mu_{k-1}(X_{A,k}) &= \min\left\{\frac{\|d_{k-1}\phi\|_X^2}{\|\phi\|_X^2} : 0 \neq \phi \in \ker d_{k-2}^*\right\} \\
&\geq \min\left\{\frac{\left(|A| - k\max_{\eta\in D(\phi)} |\widehat{1_A}(\eta)|\right)\|\phi\|_X^2}{\|\phi\|_X^2} : 0 \neq \phi \in \ker d_{k-2}^*\right\} \\
&\geq |A| - k\max_{\eta\in\widehat{G}_+} |\widehat{1_A}(\eta)|.
\end{aligned}
$$

□

4 Proof of Theorem 1.2

Let $k \geq 1$ and $0 < \epsilon < 1$ be fixed, and let $n > 2^{10}k^8/\epsilon^8$, $m = \lceil 4k^2 \log n/\epsilon^2 \rceil$. Let G be an abelian group of order n and let Ω denote the uniform probability space of all m-subsets of G. Suppose that $A \in \Omega$ satisfies $|\widehat{1_A}(\eta)| \leq \frac{\epsilon m}{k}$ for all $\eta \in \widehat{G}_+$. Then by Theorem 3.1

$$\begin{aligned}\mu_{k-1}(X_{A,k}) &\geq |A| - k \max\{|\widehat{1_A}(\eta)| : \eta \in \widehat{G}_+\} \\ &\geq |A| - k \cdot \frac{\epsilon m}{k} = (1-\epsilon)m.\end{aligned}$$

Theorem 1.2 will therefore follow from

Proposition 4.1.

$$\Pr_\Omega \left[A \in \Omega \ : \ \max_{\eta \in \widehat{G}_+} |\widehat{1_A}(\eta)| > \frac{\epsilon m}{k} \right] < \frac{6}{n}.$$

Proof. Let $\eta \in \widehat{G}_+$ be fixed and let $\lambda = \frac{\epsilon m}{k}$. Let Ω' denote the uniform probability space G^m, and for $1 \leq i \leq m$ let X_i be the random variable defined on $\omega' = (a_1, \ldots, a_m) \in \Omega'$ by $X_i(\omega') = \eta(-a_i)$. The X_i's are independent and satisfy $|X_i| = 1$. Furthermore, as $\eta \neq \chi_0$, the expectation of X_i satisfies $E_{\Omega'}[X_i] = \frac{1}{n}\sum_{x \in G} \eta(-x) = 0$. Hence, by the Chernoff bound (see e.g. Theorem A.1.16 in [3])

$$\begin{aligned}&\Pr_{\Omega'} \left[\omega' \in \Omega' \ : \ \left| \sum_{i=1}^m X_i(\omega') \right| > \lambda \right] < 2\exp\left(-\frac{\lambda^2}{2m}\right) \\ &= 2\exp\left(-\frac{\epsilon^2 m}{2k^2}\right) \leq \frac{2}{n^2}.\end{aligned} \tag{14}$$

Let $\Omega'' = \{(a_1, \ldots, a_m) \in G^m : a_i \neq a_j \text{ for } i \neq j\}$ denote the subspace of Ω' consisting of all sequences in G^m with pairwise distinct elements. Note that the assumption $n > 2^{10}k^8\epsilon^{-8}$ implies that

$$\frac{m^2}{n-m} < 1. \tag{15}$$

Combining (14) and (15) we obtain

$$\begin{aligned}&\Pr_\Omega \left[A \in \Omega : |\widehat{1_A}(\eta)| > \frac{\epsilon m}{k} \right] \\ &= \Pr_{\Omega''} \left[\omega'' \in \Omega'' : \left| \sum_{i=1}^m X_i(\omega'') \right| > \frac{\epsilon m}{k} \right] \\ &\leq \Pr_{\Omega'} \left[\omega' \in \Omega' \ : \ \left| \sum_{i=1}^m X_i(\omega') \right| > \frac{\epsilon m}{k} \right] \cdot \left(\Pr_{\Omega'}[\ \Omega''\ ]\right)^{-1} \\ &< \frac{2}{n^2} \cdot \prod_{i=1}^m \frac{n}{n-i+1} \leq \frac{2}{n^2} \cdot \left(\frac{n}{n-m}\right)^m \\ &\leq \frac{2}{n^2} \cdot \exp\left(\frac{m^2}{n-m}\right) < \frac{6}{n^2}.\end{aligned} \tag{16}$$

Using the union bound together with (16) it follows that

$$\Pr_{\Omega}\left[\max_{\eta\in\widehat{G}_+}|\widehat{1_A}(\eta)|>\frac{\epsilon m}{k}\right]<\frac{6}{n}.$$

□

Remark: As noted in the introduction, Theorem 1.2 does not hold if $|A|=o(\log|G|)$. For example, we show the following

Proposition 4.2. *Let* $1\le k\le 2^{\ell-1}$ *and suppose* $A\subset G=\mathbb{Z}_2^\ell$ *satisfies* $|A|<\ell=\log_2|G|$. *Then* $\tilde{H}^{k-1}(X_{A,k};\mathbb{R})\neq 0$, *i.e.* $\mu_{k-1}(X_{A,k})=0$.

Proof. As $|A|<\ell$, it follows that A is contained in some $(\ell-1)$-dimensional subspace W of $\mathbb{Z}_2^\ell$. Let $w_0,\ldots,w_{k-2}$ be arbitrary fixed elements of W and let $w=\sum_{i=0}^{k-2}w_i$. Let 1_W be the indicator function of A, i.e. $1_W(x)=1$ if $x\in W$, and $1_W(x)=0$ if $x\in G\setminus W$. Let $\phi\in C^{k-1}(X_{A,k})$ be the unique $(k-1)$-cochain of $X_{A,k}$ that satisfies

$$\phi([v_0,\ldots,v_{k-1}])=\begin{cases}1_W(v_{k-1}) & (v_0,\ldots,v_{k-2})=(w_0,\ldots,w_{k-2}),\\ 0 & \{w_0,\ldots,w_{k-2}\}\not\subset\{v_0,\ldots,v_{k-1}\}.\end{cases}$$

We claim that $\phi\in Z^{k-1}(X_{A,k})$. Indeed, let $\sigma=\{v_0,\ldots,v_k\}$ be a k-simplex of $X_{A,k}$. Then $\sum_{i=0}^k v_i=a\in A$. If $\{w_0,\ldots,w_{k-2}\}\not\subset\sigma$ then clearly $d_{k-1}([v_0,\ldots,v_k])=0$. Otherwise, by renumbering we may assume that $(v_0,\ldots,v_{k-2})=(w_0,\ldots,w_{k-2})$ and so $a=w+v_{k-1}+v_k$. It follows that $v_{k-1}\in W$ iff $v_k\in W$. Therefore

$$\begin{aligned}d_{k-1}\phi([v_0,\ldots,v_k])&=d_{k-1}\phi([w_0,\ldots,w_{k-2},v_{k-1},v_k])\\&=(-1)^{k-1}\phi([w_0,\ldots,w_{k-2},v_k])+(-1)^k\phi([w_0,\ldots,w_{k-2},v_{k-1}])\\&=(-1)^{k-1}\left(1_W(v_k)-1_W(v_{k-1})\right)=0.\end{aligned}$$

We next show that $\phi\notin B^{k-1}(X_{A,k})$. Assume to the contrary that $\phi=d_{k-2}\psi$ for some $\psi\in C^{k-2}(X_{A,k})$. Choose elements $w_{k-1}\in W-\{w_0,\ldots,w_{k-2}\}$ and $v_k\in G\setminus W$, and let $\tau=[w_0,\ldots,w_{k-2},w_{k-1},v_k]$. Then

$$\begin{aligned}0=\psi([\partial_{k-1}\partial_k\tau])&=d_{k-2}\psi([\partial_k\tau])=\phi([\partial_k\tau])\\&=(-1)^{k-1}\phi([w_0,\ldots,w_{k-2},v_k])+(-1)^k\phi([w_0,\ldots,w_{k-2},w_{k-1}])\\&=(-1)^{k-1}1_W(v_k)+(-1)^k1_W(w_{k-1})=(-1)^k,\end{aligned}$$

a contradiction. We have thus shown that ϕ is a nontrivial $(k-1)$-cocycle of $X_{A,k}$, hence $\tilde{H}^{k-1}(X_{A,k};\mathbb{R})\neq 0$.

□

5 Concluding Remarks

In this paper we studied the $(k-1)$-spectral gap of sum complexes $X_{A,k}$ over a finite abelian group G. Our main results include a Fourier theoretic lower bound on $\mu_{k-1}(X_{A,k})$, and a proof that for a sufficiently large constant $C(k)$, if A is a random subset of G of size at least $C(k)\log|G|$, then $X_{A,k}$ has a nearly optimal $(k-1)$-th spectral gap. Our work suggests some more questions regarding sum complexes:

- Theorem 1.2 implies that if G is an abelian group of order n, then G contains many subsets A of size $m = O_{k,\epsilon}(\log n)$ such that $\mu_{k-1}(X_{A,k}) \geq (1-\epsilon)m$. As is often the case with probabilistic existence proofs, it would be interesting to give explicit constructions for such A's. For $G = \mathbb{Z}_2^\ell$, such a construction follows from the work of Alon and Roichman. Indeed, they observed (see Proposition 4 in [2]) that by results of [1], there is an absolute constant $c > 0$ such that for any $\epsilon > 0$ and ℓ, there is an explicitly constructed $A_\ell \subset \mathbb{Z}_2^\ell$ of size
$$m \leq \frac{ck^3\ell}{\epsilon^3} = \frac{ck^3 \log_2 |G|}{\epsilon^3},$$
such that
$$|\widehat{1_{A_\ell}}(v)| = |\sum_{a\in A}(-1)^{a\cdot v}| \leq \frac{\epsilon m}{k}$$
for all $0 \neq v \in \mathbb{Z}_2^\ell$. Theorem 3.1 then implies that $\mu_{k-1}(X_{A_\ell,k}) \geq (1-\epsilon)m$.
It would be interesting to find explicit constructions with $|A| = O(\log|G|)$ for other groups G as well, in particular for the cyclic group $\mathbb{Z}_p$.

- In addition to the $(k-1)$-th spectral gap $\mu_{k-1}(X)$ of a simplicial complex X, there is different measure of the expansion of X, called the $(k-1)$-th Cheeger constant of X and denoted by $h_{k-1}(X)$. For the definition of $h_{k-1}(X)$ and a discussion of its relevance to Gromov's overlap theorem and to random complexes see [8]. In light of Theorem 1.2 it seems reasonable to conjecture that for $k \geq 1$ there exist constants $C(k), \epsilon(k) > 0$ such that random sum complexes $X_{A,k}$ with $|A| = C(k)\log|G|$ satisfy $h_{k-1}(X_{A,k}) \geq \epsilon(k)$ asymptotically almost surely as $|G| \to \infty$.

- Consider the following non-abelian version of sum complexes. Let G be a finite group of order n and let $A \subset G$. For $1 \leq i \leq k+1$ let V_i be the 0-dimensional complex on the set $G \times \{i\}$, and let $T_{n,k}$ be the join $V_1 * \cdots * V_{k+1}$. The complex $R_{A,k}$ is obtained by taking the $(k-1)$-skeleton of $T_{n,k}$, together with all k-simplices $\sigma = \{(x_1, 1), \ldots, (x_{k+1}, k+1)\} \in T_{n,k}$ such that $x_1 \cdots x_{k+1} \in A$. One may ask whether there is an analogue of Theorem 1.2 for the complexes $R_{A,k}$, i.e. is there a constant $c_1(k,\epsilon) > 0$ such that if A is a random subset of G of size $m = \lceil c_1(k,\epsilon)\log n\rceil$, then a.a.s. $\mu_{k-1}(R_{A,k}) > (1-\epsilon)m$.

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