On families of faces in discrete cubes

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Abstract

Let $\mathcal{F}$ be a family of $n-k$ -dimensional faces of the discrete cube $\{0,1\}^n$ such that for all $F \in \mathcal{F}$, $\quad F \not\subset \bigcup\{F' : F \neq F' \in \mathcal{F}\}$. It is shown that if $n \geq n_0(k)$ then $|\mathcal{F}| \leq \binom{n}{k}$. This was conjectured by Aharoni and Holzman and is the case $m=2$ of a more general result on faces of $\{0,\ldots,m-1\}^n$.

1 Introduction

Let $M = \{0, \ldots, m-1\}$ and $M^n = \{(x_1, \ldots, x_n) : x_i \in M\}$.
An $n-k$ -dimensional face of M^n is a subset of the form
$A = \{(x_1, \ldots, x_n) \in M^n : x_i = \alpha_i \ for\ all\ i \in I\}$, where I is a fixed k-subset of $[n]$, and the α_i's $i \in I$ are some fixed elements of M. The face A shall also be described by the vector $a \in (M \cup *)^n$, where $a_i = \alpha_i$ for $i \in I$, and $a_i = *$ otherwise.
Let $f_m(n,k)$ be the maximal t for which there exists a family $A_1, \ldots, A_t$ of $n-k$ -dimensional faces of M^n, such that for all $1 \leq i \leq t$, $A_i \not\subset \bigcup_{j \neq i} A_j$.

Example: for each $\{i_1 < \cdots < i_k\} \subset [n]$ and $\alpha = (\alpha_1, \ldots, \alpha_k) \in (M - \{0\})^k$, let $A_{I,\alpha} = \{x \in M^n : x_{i_j} = \alpha_j \ \ for\ all \ \ 1 \leq j \leq k\}$.
Clearly $A_{I,\alpha} \not\subset \bigcup_{(J,\beta) \neq (I,\alpha)} A_{J,\beta}$, and thus $f_m(n,k) \geq (m-1)^k \binom{n}{k}$.

Aharoni and Holzman [1] proved that $f_2(n,k) \leq \sum_{i=0}^{k} \binom{n}{i}$ and that $f_2(n,2) = \binom{n}{2}$ for $n \geq 5$. They conjectured that $f_2(n,k) = \binom{n}{k}$ for $n \geq n_0(k)$.
Here we show

Theorem 1:

$$f_m(n,k) \leq (m-1)^k \binom{n}{k} \frac{m^n}{\sum_{j=k}^{n} (m-1)^j \binom{n}{j}}$$

which implies

Corollary 2 : $f_m(n,k) = (m-1)^k \binom{n}{k} \quad$ for $\quad n \geq 10k$.

2 Proofs

Proof of Theorem 1: Let $t = f_m(n,k)$ and suppose $A_1, \ldots, A_t$ are $n-k$ -dimensional faces of M^n, such that for all $1 \leq i \leq t$, $A_i \not\subset \bigcup_{j \neq i} A_j$. Let $a_i = (a_{i1}, \ldots, a_{in}) \in (M \cup *)^n$ be the vector which describes A_i, and let $\alpha_i = (\alpha_{i1}, \ldots, \alpha_{in}) \in A_i - \bigcup_{j \neq i} A_j \subset M^n$.
Denote $\mathcal{U} = \{(u_1, \ldots, u_{n-k}) : 1 \leq u_i \neq u_j \leq n\}$, $|\mathcal{U}| = \frac{n!}{k!}$. For each $u \in \mathcal{U}$ and $1 \leq l \leq t$ let $p(l,u) = \min\{j : a_{lu_j} \neq *\}$. If $a_{lu_j} = *$ for all $1 \leq j \leq n-k$

set $p(l,u) = n-k+1$.

A *configuration* $\beta = (\beta_1, \ldots, \beta_{n-k}) \in M^{n-k}$ is *compatible* with a pair $(l,u) \in [t] \times \mathcal{U}$, if $\alpha_{lu_j} = \beta_j$ for all $j \geq p(l,u)$.
For $u \in \mathcal{U}$ and $\beta \in M^{n-k}$ we denote by $L(u,\beta)$, the set of all $1 \leq l \leq t$ for which β is compatible with (l,u).

Claim 3: $|L(u,\beta)| \leq m^k$ for all u and β.

Proof: For each $l \in L(u,\beta)$ consider the vector $\gamma_l = (\alpha_{li} : i \notin \{u_1, \ldots, u_{n-k}\}) \in M^k$. It suffices to show that if $l \neq l'$ are in $L(u,\beta)$ then $\gamma_l \neq \gamma_{l'}$.
Suppose, say, that $p(l,u) \geq p(l',u)$. By assumption $\alpha_{l'} \notin A_l$, so there exists an $1 \leq i \leq n$ such that $a_{li} \neq *$ and $\alpha_{l'i} \neq a_{li} = \alpha_{li}$. Now for $1 \leq j \leq n-k$, either $j < p(l,u)$ and then $a_{lu_j} = *$, or $j \geq p(l,u) \geq p(l',u)$ and then $\alpha_{lu_j} = \beta_j = \alpha_{l'u_j}$. Thus $a_{li} \neq *$ and $\alpha_{l'i} \neq \alpha_{li}$ together imply that $i \notin \{u_1, \ldots, u_{n-k}\}$ and so $\gamma_l \neq \gamma_{l'}$.

$\square$

Now $\sum_{u,\beta} |L(u,\beta)|$ may be evaluated as follows: For a fixed $1 \leq l \leq t$, we count the number of pairs (u,β) for which β is compatible with (l,u).
For $u \in \mathcal{U}$, the number of $\beta \in M^{n-k}$ which are compatible with (l,u) is clearly $m^{p(l,u)-1}$. For any $1 \leq j \leq n-k+1$, the number of $u \in \mathcal{U}$ such that $p(l,u) = j$ is

$$(n-k)\cdots(n-k-(j-1)+1)\cdot k \cdot (n-j)\cdots(k+1) = \frac{(n-k)!}{(n-k-(j-1))!} \cdot \frac{(n-j)!}{(k-1)!}$$

(To form such a u, we first choose a sequence of $j-1$ $*$'s from a_l, then a non-$*$, and then complete u by a sequence of $n-k-j$ elements out of the remaining $n-j$ elements of a_l).
Therefore, the number of pairs (u,β) such that β is compatible with (l,u) is

$$\sum_{j=1}^{n-k+1} m^{j-1} \frac{(n-k)!}{(n-k-(j-1))!} \cdot \frac{(n-j)!}{(k-1)!} = (n-k)! \sum_{j=0}^{n-k} m^j \binom{n-1-j}{k-1}$$

combining this with Claim 3 we get:

$$(\frac{n!}{k!}m^{n-k})\cdot m^k \geq \sum_{u,\beta}|L(u,\beta)| = t(n-k)!\sum_{j=0}^{n-k} m^j\binom{n-1-j}{k-1}$$

and so

$$f_m(n,k) = t \leq \binom{n}{k}\frac{m^n}{\sum_{j=0}^{n-k} m^j\binom{n-1-j}{k-1}} \quad .$$

It remains to show that

$$(m-1)^k\sum_{j=0}^{n-k} m^j\binom{n-1-j}{k-1} = \sum_{j=k}^{n}(m-1)^j\binom{n}{j} \quad .$$

Now $\binom{n-1-j}{k-1}(m-1)^k m^j$ is the number of vectors $c \in M^n$ whose k-th non-zero entry occurs in place $n-j$. Hence $\sum_{j=0}^{n-k}\binom{n-1-j}{k-1}(m-1)^k m^j$ is simply the number of $c \in M^n$ with $\geq k$ non-zero entries, which in turn equals $\sum_{j=k}^{n}(m-1)^j\binom{n}{j}$.

□

Proof of Corollary 2: By Theorem 1,

$$f_m(n,k) - (m-1)^k\binom{n}{k} \leq \frac{(m-1)^k\binom{n}{k}\sum_{j=0}^{k-1}(m-1)^j\binom{n}{j}}{m^n - \sum_{j=0}^{k-1}(m-1)^j\binom{n}{j}} \quad .$$

When $n \geq 4k$ the right hand side is at most

$$\frac{2k\binom{n}{k}^2(m-1)^{2k}}{m^n} \leq \frac{2k\binom{n}{k}^2}{2^n}$$

which by standard estimates is less then 1 when $n \geq 10k$.

□

References

[1] R. Aharoni and R. Holzman, Private Communication.

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