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A Geometrical Problem Arising in a Signal Restoration Algorithm

Nir Cohen and Roy Meshulam

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ABSTRACT. *Let* $\mathbb{Z}_{2N} = \{0, \ldots, 2N-1\}$ *denote the group of integers modulo* $2N$, *and let* L *be the space of all real functions on* $\mathbb{Z}_{2N}$ *which are supported on* $\{0, \ldots, N-1\}$. *The spectral phase of a function* $f : \mathbb{Z}_{2N} \to \mathbb{R}$ *is given by* $\phi_f(k) = \arg \widehat{f}(k)$ *for* $k \in \mathbb{Z}_{2N}$, *where* $\widehat{f}$ *denotes the discrete Fourier transform of* f.

For a fixed $s \in L$ *let* K_s *denote the cone of all* $f : \mathbb{Z}_{2N} \to \mathbb{R}$ *which satisfy* $\phi_f \equiv \phi_s$, *and let* M_s *be its linear span. The angle* α_s *between* M_s *and* L *determines the convergence rate of the signal restoration from phase algorithm of Levi and Stark [3]. Here we prove the following conjectures of Urieli et al. [7] who verified them for the* $N \leq 3$ *case:*

1. $\alpha(M_s, L) \leq \pi/4$ *for a generic* $s \in L$.

2. If $s \in L$ *is geometric, i.e.,* $s(j) = q^j$ *for* $0 \leq j \leq N-1$ *where* $\pm 1 \neq q \in \mathbb{R}$, *then* $\alpha(M_s, L) = \pi/4$.

1. Introduction

Let $\mathbb{Z}_{2N}$ denote the group of integers modulo $2N$ ($N \geq 2$), and let $L_{\mathbb{C}}(\mathbb{Z}_{2N})$ ($L_{\mathbb{R}}(\mathbb{Z}_{2N})$) denote the space of complex (real) valued functions on $\mathbb{Z}_{2N}$ with the usual inner product $\langle f, g\rangle = \sum_{x \in \mathbb{Z}_{2N}} f(x)\overline{g(x)}$. The Fourier transform $F : L_{\mathbb{C}}(\mathbb{Z}_{2N}) \to L_{\mathbb{C}}(\mathbb{Z}_{2N})$ is given by $F(f)(y) = \widehat{f}(y) = \sum_{x \in \mathbb{Z}_{2N}} f(x)\omega^{-xy}$ where $\omega = \exp(\pi i/N)$. The spectral phase of $f \in L_{\mathbb{R}}(\mathbb{Z}_{2N})$ is given by $\phi_f(k) = \arg \widehat{f}(k)$ for $k \in \mathbb{Z}_{2N}$.

A natural and important problem which arises in diverse applications is the reconstruction of a Fourier transform pair $f, F(f)$ from partial data on either or both functions. See Barakat and Newsam's papers [1, 2] for a study of various reconstruction algorithms and their analysis. One particular instance of the general reconstruction problem which is commonly encountered in image processing is the retrieval of a signal $f \in L_{\mathbb{R}}(\mathbb{Z}_{2N})$ from its spectral phase ϕ_f.

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Denote Supp $f = \{x \in \mathbb{Z}_{2N} : f(x) \neq 0\}$, and let $L = \{s \in L_{\mathbb{R}}(\mathbb{Z}_{2N}) : \text{Supp}\, s \subset \{0, \dots, N-1\}\}$. For a function $s \in L$, let K_s denote the cone of all $f \in L_{\mathbb{R}}(\mathbb{Z}_{2N})$ such that $\phi_f \equiv \phi_s$, and let M_s be its linear span. Clearly $s \in K_s \cap L$.

Levi and Stark [3] (see also [6]) noted that for a dense Zariski open subset of functions s in L, the intersection $K_s \cap L$ is exactly Span$\{s\}$ and may be efficiently reconstructed from the spectral phase ϕ_s. The projection on convex sets (POCS) algorithm developed in [3] applies successive alternate projections on K_s and L which eventually converge to Span$\{s\}$ from any given initial state f_0. Let α_s denote the angle between M_s and L. The convergence rate of the POCS algorithm is determined by α_s as follows: Let f_k denote the signal obtained after the kth two-step iteration of the POCS algorithm. It is easy to show that

$$\|f_{k+1} - f_k\| \leq \cos^2(\alpha_s)\, \|f_k - f_{k-1}\| \tag{1.1}$$

and that $\cos^2(\alpha_s)$ is the best constant for which (1.1) holds, assuming an arbitrary initial state f_0.

In a recent paper, Urieli et al. [7] studied the relation between the spatial profile of s and the angle α_s. Based on extensive empirical evidence, they conjectured that $\alpha_s \leq \pi/4$ and that the upper bound is attained when s has an exponential profile. At the other extreme, they showed that when s is close to symmetric, the angle α_s approaches zero (for a symmetric signal the problem is ill-posed).

While our main concern in this article is the behavior of the angle α_s, it should be noted that in practice the performance of the POCS algorithm is more complex, and is not determined by (1.1) alone. Indeed, a physically realistic model for the amplitude retrieval problem must incorporate model uncertainty and measurement noise. As noted in [1], an exact analysis of noise effects in iterative reconstruction algorithms is difficult. We can however offer the following qualitative remarks concerning the stability of the POCS algorithm with respect to errors: The generic positivity of α_s combined with the estimate (1.1) guarantees strict contractivity of the POCS, hence convergence to the (essentially unique) solution, for the nominal problem. The continuity of α_s w.r.t. s ([7, Theorem 3]) guarantees the same for the perturbed model. To handle measurement noise, one can define at each iteration a "local angle" which dictates the contractivity at that step. The local angle, which is bounded below by α_s, is continuous w.r.t. the location of the given iterate ([7, Lemma C1]). This guarantees the stability of the convergence process at the presence of relatively small measurement errors. Simulations carried out in [7] appear to support this conclusion.

Returning to the nominal reconstruction problem, our main results (Theorems 1 and 2) are proofs of the above-mentioned conjectures of Urieli et al. [7]. We start with some preliminaries.

The Fourier transform satisfies the Parseval identity $\langle \widehat{f}, \widehat{g} \rangle = 2N\langle f, g \rangle$, and the inversion formula $F^{-1}(g)(m) = (2N)^{-1}\widehat{g}(-m)$. Note that $f \in L_{\mathbb{R}}(\mathbb{Z}_{2N})$ iff $\widehat{f}(-m) = \overline{\widehat{f}(m)}$ for all $m \in \mathbb{Z}_{2N}$. The convolution of $f, g \in L_{\mathbb{C}}(\mathbb{Z}_{2N})$ is given by $f * g(x) = \sum_{y \in \mathbb{Z}_{2N}} f(y)g(x-y)$ and satisfies $\widehat{f * g}(x) = \widehat{f}(x)\widehat{g}(x)$.

Let $\Lambda_{\mathbb{C}} = \{\lambda \in L_{\mathbb{C}}(\mathbb{Z}_{2N}) : \lambda(j) = \lambda(-j) \text{ for all } j \in \mathbb{Z}_{2N}\}$ and let $\Lambda_{\mathbb{R}} = \Lambda_{\mathbb{C}} \cap L_{\mathbb{R}}(\mathbb{Z}_{2N})$. Clearly, $F(\Lambda_{\mathbb{C}}) = \Lambda_{\mathbb{C}}$. For a fixed $s \in L$ let $W_s = \widehat{s} \cdot \Lambda_{\mathbb{R}} = \{\lambda\widehat{s} : \lambda \in \Lambda_{\mathbb{R}}\}$, and $W_s^+ = \{\lambda\widehat{s} : 0 \leq \lambda \in \Lambda_{\mathbb{R}}\}$. Clearly, $F^{-1}(W_s^+) = K_s$ and $F^{-1}(W_s) = M_s$. Since $\lambda\widehat{s}(-m) = \overline{\lambda\widehat{s}(m)}$ for all $\lambda \in \Lambda_{\mathbb{R}}$ and $m \in \mathbb{Z}_{2N}$, it follows that $M_s = F^{-1}(W_s) \subset L_{\mathbb{R}}(\mathbb{Z}_{2N})$ and $\dim M_s = \dim W_s = |\text{Supp}\, \widehat{s} \cap \{0, \dots, N\}|$.

A function $s \in L$ will be called *generic* if $M_s \cap L = \text{Span}\{s\}$. For $N \leq m \leq 2N-1$ and $1 \leq j \leq N$ let

$$D_s(m, j) = \begin{cases} s(m-j) + s(m+j) & \text{if } 1 \leq j \leq N-1 \\ s(m-N) & \text{if } j = N \end{cases}$$

Claim 1.

If $s \in L$ satisfies $\det D_s \neq 0$*, then s is generic.*

Proof. Let $N \le m \le 2N-1$. Then for $\lambda \in \Lambda_{\mathbb{R}}$

$$F^{-1}(\lambda\widehat{s})(m) = \frac{1}{2N}\widehat{\lambda} * s(m) = \frac{1}{2N}\sum_{j=0}^{2N-1} s(m-j)\widehat{\lambda}(j) = \frac{1}{2N}\sum_{j=1}^{N} D_s(m,j)\widehat{\lambda}(j)\,.$$

Suppose D_s is non-singular. If $\lambda \in \Lambda_{\mathbb{R}}$ satisfies $F^{-1}(\lambda\widehat{s}) \in L$, then $F^{-1}(\lambda\widehat{s})(m) = 0$ for all $N \le m \le 2N-1$, and hence, $\widehat{\lambda}(j) = 0$ for all $1 \le j \le N$. It follows that λ is constant, and therefore $F^{-1}(\lambda\widehat{s}) \in \text{Span}\{s\}$. □

Claim 1 implies that the set of generic functions contains $S = \{s \in L \ : \ \det D_s \neq 0\}$ which is a Zariski open dense subset of L. Similar genericity criteria appear in [4] and [5]. For $s \in S$ the angle between M_s and L is given by

$$\alpha_s = \min\{\alpha(f,g) \ : \ 0 \neq f \in M_s\,,\ 0 \neq g \in L\,,\ \langle f,s\rangle = \langle g,s\rangle = 0\}$$

where $\alpha(f,g)$ denotes the angle between f and g.

Theorems 1 and 2 were conjectured by Urieli et al. [7] and proved by them for $N \le 3$.

Theorem 1.

$\alpha_s \le \pi/4$ for all $s \in S$.

Our main result deals with a case of equality in Theorem 1. A function $s \in L$ is *geometric with parameter q* if $s(j) = q^j$ for all $0 \le j \le N-1$.

Theorem 2.

If $s \in L$ is geometric with parameter $\pm 1 \neq q \in \mathbb{R}$, then $s \in S$ and $\alpha_s = \pi/4$.

Theorems 1 and 2 are proved in Sections 2 and 3.

2. The Upper Bound

For $t \in \{0,1\}$ let $E_t = \{0 \le j \le N \ : \ j \equiv t \pmod 2\}$. We shall need the following technical observation:

Claim 2.

Suppose $0 \neq s \in L$ satisfies $|\text{Supp}\,\widehat{s} \cap E_t| \le 1$ for $t = 0, 1$. Then $N \le 3$ and s is a multiple of one of the following functions (written as vectors in $\mathbb{R}^{2N}$):

1. *$N = 2$: $(1,1,0,0)$ or $(1,-1,0,0)$.*
2. *$N = 3$: $(1,0,-1,0,0,0)$.*

Proof. By assumption there exist $0 \le 2k, 2l+1 \le N$ and $\alpha, \beta \in \mathbb{C}$ such that for all $x \in \mathbb{Z}_{2N}$

$$s(x) = \alpha\omega^{-2kx} + \overline{\alpha}\omega^{2kx} + \beta\omega^{-(2l+1)x} + \overline{\beta}\omega^{(2l+1)x}\,. \tag{2.1}$$

Let $0 \le j \le N-1$, then

$$0 = s(j+N) = \alpha\omega^{-2kj} + \overline{\alpha}\omega^{2kj} - \beta\omega^{-(2l+1)j} - \overline{\beta}\omega^{(2l+1)j}$$

hence, by (2.1)

$$s(j) = 2\left(\alpha\omega^{-2kj} + \overline{\alpha}\omega^{2kj}\right)\,. \tag{2.2}$$

We consider two cases:

1. If $2k \equiv 0 \pmod N$, then $s(j) = 2(\alpha+\overline{\alpha})\epsilon^j$ for a fixed $\epsilon \in \{\pm 1\}$ and for all $0 \le j \le N-1$. It follows that all odd m

$$\widehat{s}(m) = \frac{4(\alpha+\overline{\alpha})}{1-\epsilon\omega^{-m}} \neq 0\,.$$

The assumption $|\mathrm{Supp}\,\widehat{s} \cap E_1| \le 1$ then implies that $N = 2$ and s is a multiple of $(1, \epsilon, 0, 0)$.

2. Suppose $2k \not\equiv 0 \pmod N$. Let $2l + 1 \ne m \in E_1$, then by (2.2)

$$0 = \widehat{s}(m) = \frac{4\alpha}{1 - \omega^{-(2k+m)}} + \frac{4\overline{\alpha}}{1 - \omega^{2k-m}}$$

hence,

$$\left|1 - \omega^{-(2k+m)}\right| = \left|1 - \omega^{2k-m}\right| . \tag{2.3}$$

(2.3) together with $2k \not\equiv 0 \pmod N$ imply $m = N$. Since this holds for all odd $m \ne 2l + 1$ in $\{0, \dots, N\}$, it follows that $N \le 3$. The cases $N = 2$ and $N = 3$, $k = 0$ are covered by 1. The remaining possibility is $N = 3$, $k = 1$ which implies $s(j) = 2(\alpha\omega^{-2j} + \overline{\alpha}\omega^{2j})$ for $0 \le j \le 2$. It can be checked that $\widehat{s}(1) \ne 0$; hence,

$$\widehat{s}(3) = \frac{4\alpha}{1 - \omega} + \frac{4\overline{\alpha}}{1 - \omega^{-1}} = 0 .$$

Therefore, $\alpha/\overline{\alpha} = \omega$ and $s(j) = 2\overline{\alpha}(\omega^{-2j+1} + \omega^{2j})$ for $0 \le j \le 2$. It follows that s is a multiple of $(1, 0, -1, 0, 0, 0)$. □

By checking that $\det D_s = 0$ for each of the three exceptional cases in Claim 2 we obtain the following

Corollary 1.
For any $s \in S$ there exists a $t \in \{0, 1\}$ such that $|\mathrm{Supp}\,\widehat{s} \cap E_t| \ge 2$. □

Proof of Theorem 1. Choose $t \in \{0, 1\}$ such that $A_t = \mathrm{Supp}\,\widehat{s} \cap E_t$ satisfies $|A_t| \ge 2$. Since

$$\langle \lambda\widehat{s}, \widehat{s}\rangle = \lambda(0)\,|\widehat{s}(0)|^2 + 2\sum_{m=1}^{N-1} \lambda(m)\,|\widehat{s}(m)|^2 + \lambda(N)\,|\widehat{s}(N)|^2$$

it follows that there exists a $0 \ne \lambda \in \Lambda_{\mathbb{R}}$ such that $\mathrm{Supp}\,\lambda \cap \{0, \dots, N\} \subset A_t$ and $\langle \lambda\widehat{s}, \widehat{s}\rangle = 0$. Let $g = \lambda\widehat{s} \in W_s - \{0\}$. Since $g(2j + t + 1) = 0$ for all $j \in \mathbb{Z}_{2N}$, it follows that

$$\begin{aligned}
\widehat{g}(m + N) &= \sum_{j=0}^{N-1} g(2j + t)\omega^{-(2j+t)(m+N)} \\
&= (-1)^t \sum_{j=0}^{N-1} g(2j + t)\omega^{-(2j+t)m} = (-1)^t\widehat{g}(m) .
\end{aligned} \tag{2.4}$$

Let $f = F^{-1}(g) \in M_s - \{0\}$, and let $h \in L$ be given by $h(m) = f(m)$ for $0 \le m \le N - 1$ and $h(m) = 0$ otherwise. (2.4) implies that $|f(m)| = |f(m + N)|$ for all $m \in \mathbb{Z}_{2N}$, hence, $\langle h, f\rangle = \langle h, h\rangle = \frac{1}{2}\langle f, f\rangle$. Let β denote the angle between f and h, then

$$\cos\beta = \frac{\langle h, f\rangle}{\|f\|\,\|h\|} = \frac{1}{\sqrt{2}} .$$

Now

$$\langle h, s\rangle = \langle f, s\rangle = \left\langle F^{-1}(g), F^{-1}(\widehat{s})\right\rangle = \frac{1}{2N}\langle g, \widehat{s}\rangle = 0$$

hence, both $f \in M_s - \{0\}$ and $h \in L - \{0\}$ are orthogonal to $M_s \cap L$. It follows that $\alpha_s \le \beta = \pi/4$.

Remark. For functions $s \in S$ which satisfy the additional condition $\mathrm{Supp}\,\widehat{s} = \mathbb{Z}_{2N}$ (or equivalently $\dim M_s = N + 1$), Claim 2 and Corollary 1 are of course superfluous. □

3. The Geometric Case

Let $s \in L$ be geometric with parameter $q \in \mathbb{R}$.

Claim 3.

$$\det D_s = \begin{cases} \left(q^N - 1\right)\left(q^{2N} - 1\right)^{\frac{N}{2}-1} & N \text{ even} \\ \left(q^{2N} - 1\right)^{\frac{N-1}{2}} & N \text{ odd} \end{cases} \tag{3.1}$$

Proof. It is convenient to re-index the rows of D_s by defining $C_q(i, j) = D_s(i + N - 1, j)$ for $1 \le i, j \le N$. More explicitly, let A_q, B_q be the $N \times N$ matrices given by

$$A_q(i, j) = \begin{cases} q^{N-1+i-j} & 1 \le i \le j \le N \\ 0 & \text{otherwise} \end{cases}$$
$$B_q(i, j) = \begin{cases} q^{i+j-N-1} & N + 1 \le i + j \text{ and } j < N \\ 0 & \text{otherwise} \end{cases}$$

then $C_q = A_q + B_q$.

For $1 \le i \le N$ let $C_q(i)$ denote the ith row of C_q. We prove (3.1) for N odd: Let $\zeta = \omega^l$ be any $2N$th root of 1. The rows $\{C_\zeta(i) \ : \ 1 \le i \le \frac{N+1}{2}\}$ are clearly linearly independent. A routine computation (we omit the details) shows that for $\frac{N+3}{2} \le k \le N$

$$C_\zeta(k) = \zeta^{-(N-1)}\left(C_\zeta(N-k+1) + \left(\zeta - \zeta^{-1}\right)\sum_{j=2}^{k-\frac{N+1}{2}} \zeta^{j-2} C_\zeta(N-k+j)\right) + \left(\zeta - \zeta^{-1}\right)\left(1 + \zeta^{N-1}\right)^{-1} \zeta^{k-\frac{N+3}{2}} C_\zeta\left(\frac{N+1}{2}\right),$$

therefore, $C_\zeta(k) \in \operatorname{Span}\{C_\zeta(i) \ : \ 1 \le i \le \frac{N+1}{2}\}$. It follows that $\operatorname{rank} C_\zeta = \frac{N+1}{2}$, hence, the polynomial $P(q) = \det C_q$ is divisible by $(q - \zeta)^{N - \operatorname{rank} C_\zeta} = (q - \omega^l)^{\frac{N-1}{2}}$. Since $\deg P(q) = N(N-1)$ it follows that

$$\det D_s = P(q) = \prod_{l=0}^{2N-1}\left(q - \omega^l\right)^{\frac{N-1}{2}} = \left(q^{2N} - 1\right)^{\frac{N-1}{2}}.$$

The proof of (3.1) for N even is similar. □

Assume now that $q \ne \pm 1$, then $s \in S$ by Claim 3. Define an hermitian form B on $\Lambda_{\mathbb{C}} \times \Lambda_{\mathbb{C}}$ by

$$B(u, v) = \sum_{m=N}^{2N-1} u * s(m)\, \overline{v * s(m)} - \sum_{m=0}^{N-1} u * s(m)\, \overline{v * s(m)}.$$

For $i \in \mathbb{Z}_{2N}$ let $e_i \in L_{\mathbb{C}}(\mathbb{Z}_{2N})$ be given by $e_i(k) = 1$ if $i = k$ and 0 otherwise. The main ingredient of the proof of Theorem 2 is the following identity.

Claim 4.

$$\frac{1 - q^{2N}}{1 - q^2} B(u, v) = B\left(u, e_N\right)\overline{B\left(v, e_N\right)} - B\left(u, e_0\right)\overline{B\left(v, e_0\right)}. \tag{3.2}$$

Proof. Let $\theta \in L_{\mathbb{C}}(\mathbb{Z}_{2N})$ be given by

$$\theta(j) = \begin{cases} -1 & \text{if } 0 \le j \le N-1 \\ 1 & \text{if } N \le j \le 2N-1 . \end{cases}$$

Note that

$$\widehat{\theta}(m) = \begin{cases} \frac{-4}{1-\omega^{-m}} & m \text{ odd} \\ 0 & m \text{ even} \end{cases}$$

and

$$\widehat{s}(m) = \frac{1-(-1)^m q^N}{1-q\omega^{-m}} .$$

Denoting $A_N = \{(k,l) \ : \ 0 \le k, l \le 2N-1 \text{ and } k \not\equiv l \pmod 2\}$ we obtain

$$\begin{aligned} B(u,v) &= \langle \theta(u*s), v*s\rangle = \frac{1}{2N}\langle (\theta(u*s))\widehat{\ }, \ (v*s)\widehat{\ }\rangle \\ &= \frac{1}{4N^2}\left\langle \widehat{\theta} * (\widehat{us}), \widehat{vs}\right\rangle = \frac{1}{4N^2}\sum_{(k,l)\in A_N} \widehat{\theta}(l-k)\widehat{s}(k)\,\overline{\widehat{s}(l)}\,\widehat{u}(k)\,\overline{\widehat{v}(l)} \\ &= \frac{-1}{4N^2}\sum_{(k,l)\in A_N} \frac{4\left(1-q^{2N}\right)\widehat{u}(k)\overline{\widehat{v}(l)}}{\left(1-\omega^{k-l}\right)\left(1-q\omega^{-k}\right)\left(1-q\omega^{l}\right)} . \end{aligned} \tag{3.3}$$

To evaluate the right-hand side of (3.2) we note that $\widehat{e_N}(m) = (-1)^m$, $B(u,e_0) = -\langle u*s, s\rangle$ and $B(u,e_N) = \langle u*s, e_N*s\rangle$. Using the Parseval identity it follows that

$$\begin{aligned} & B(u,e_N)\,\overline{B(v,e_N)} - B(u,e_0)\,\overline{B(v,e_0)} \\ &= \langle u*s, e_N*s\rangle\,\overline{\langle v*s, e_N*s\rangle} - \langle u*s,s\rangle\overline{\langle v*s,s\rangle} \\ &= \frac{1}{4N^2}\left(\langle\widehat{us},\widehat{e_N s}\rangle\,\overline{\langle\widehat{vs},\widehat{e_N s}\rangle} - \langle\widehat{us},\widehat{s}\rangle\,\overline{\langle\widehat{vs},\widehat{s}\rangle}\right) \\ &= \frac{1}{4N^2}\sum_{k,l=0}^{2N-1}\left((-1)^{k+l}-1\right)|\widehat{s}(k)|^2\,|\widehat{s}(l)|^2\,\widehat{u}(k)\overline{\widehat{v}(l)} \\ &= \frac{-1}{4N^2}\sum_{(k,l)\in A_N}\frac{2\left(1-q^{2N}\right)^2\widehat{u}(k)\overline{\widehat{v}(l)}}{\left|1-q\omega^{-k}\right|^2\left|1-q\omega^{-l}\right|^2} . \end{aligned} \tag{3.4}$$

Now (3.3) and (3.4) imply that (3.2) is equivalent to

$$\begin{aligned} &\frac{2}{1-q^2}\sum_{(k,l)\in A_N}\frac{\widehat{u}(k)\overline{\widehat{v}(l)}}{\left(1-\omega^{k-l}\right)\left(1-q\omega^{-k}\right)\left(1-q\omega^{l}\right)} \\ &= \sum_{(k,l)\in A_N}\frac{\widehat{u}(k)\overline{\widehat{v}(l)}}{\left|1-q\omega^{-k}\right|^2\left|1-q\omega^{-l}\right|^2} . \end{aligned} \tag{3.5}$$

Since $\widehat{u}(k) = \widehat{u}(-k)$ and $\widehat{v}(l) = \widehat{v}(-l)$, it suffices to show that the sum of the coefficients of $\widehat{u}(k)\overline{\widehat{v}(l)}$ and $\widehat{u}(-k)\overline{\widehat{v}(-l)}$ is the same on both sides of (3.5). This follows from the identity:

$$\begin{aligned} &\frac{1}{\left(1-\omega^{k-l}\right)\left(1-q\omega^{-k}\right)\left(1-q\omega^{l}\right)} + \frac{1}{\left(1-\omega^{l-k}\right)\left(1-q\omega^{k}\right)\left(1-q\omega^{-l}\right)} \\ &= \frac{1-q^2}{\left|1-q\omega^{-k}\right|^2\left|1-q\omega^{-l}\right|^2} . \qquad \square \end{aligned}$$

Claim 4 now implies the following:

Claim 5.

If $u \in \Lambda_{\mathbb{C}}$ *satisfies* $\langle u * s, s \rangle = 0$, *then*

$$\sum_{m=N}^{2N-1} |u * s(m)|^2 \geq \sum_{m=0}^{N-1} |u * s(m)|^2 .$$

Proof. Since $B(u, e_0) = -\langle u * s, s \rangle = 0$ we obtain by Claim 4:

$$\sum_{m=N}^{2N-1} |u * s(m)|^2 - \sum_{m=0}^{N-1} |u * s(m)|^2 = B(u, u)$$
$$= \frac{1-q^2}{1-q^{2N}} \left(|B(u, e_N)|^2 - |B(u, e_0)|^2 \right) = \frac{1-q^2}{1-q^{2N}} |B(u, e_N)|^2 \geq 0 . \qquad \square$$

Proof of Theorem 2. By Theorem 1 it suffices to show $\alpha_s \geq \pi/4$. Let $0 \neq f \in M_s$ be orthogonal to s and let $0 \neq h \in L$. Write $f = \widehat{\lambda} * s$ where $\lambda \in \Lambda_{\mathbb{R}}$. Since $\widehat{\lambda} \in \Lambda_{\mathbb{C}}$, Claim 5 implies $\sum_{m=N}^{2N-1} |f(m)|^2 \geq \sum_{m=0}^{N-1} |f(m)|^2$. Let β denote the angle between f and h, then

$$\cos\beta = \frac{|\langle h, f \rangle|}{\|f\| \, \|h\|} = \frac{\left|\sum_{m=0}^{N-1} f(m)h(m)\right|}{\|f\| \, \|h\|} \leq \frac{\left(\sum_{m=0}^{N-1} |f(m)|^2\right)^{\frac{1}{2}} \|h\|}{\|f\| \, \|h\|} \leq \frac{1}{\sqrt{2}} .$$

It follows that $\beta \geq \pi/4$. $\square$

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Mathematics Department, University of Campinas, CP 6065 CEP 13081, Campinas SP, Brazil
e-mail: nir@ime.unicamp.br

Department of Mathematics, Technion, Haifa 32000, Israel
e-mail: meshulam@leeor.technion.ac.il