Discrete Comput. Geom. manuscript No.
(will be inserted by the editor)

Pach's selection theorem does not admit a topological extension

Imre Bárány · **Roy Meshulam** · **Eran Nevo** ·
Martin Tancer

Received: date / Accepted: date

Abstract Let $U_1,\dots,U_{d+1}$ be n-element sets in $\mathbb{R}^d$. Pach's selection theorem says that there exist subsets $Z_1 \subset U_1,\dots,Z_{d+1} \subset U_{d+1}$ and a point $u \in \mathbb{R}^d$ such that each $|Z_i| \geq c_1(d)n$ and $u \in \mathrm{conv}\{z_1,\dots,z_{d+1}\}$ for every choice of $z_1 \in Z_1,\dots,z_{d+1} \in Z_{d+1}$. Here we show that this theorem does not admit a topological extension with linear size sets Z_i. However, there is a topological extension where each $|Z_i|$ is of order $(\log n)^{1/d}$.

Keywords Pach's Selection Theorem · Gromov's Overlap Theorem

Mathematics Subject Classification (2000) MSC 52A35 · MSC 52C99

1 Introduction

Pach's homogeneous selection theorem is the following key result in discrete geometry.

Theorem 1.1 (Pach [12]). *For $d \geq 1$ there exists a constant $c_1(d) > 0$ such that the following holds. For any n-element sets $U_1,\dots,U_{d+1}$ in $\mathbb{R}^d$, there exist subsets*

Imre Bárány
Rényi Institute, Hungarian Academy of Sciences, POB 127, 1364 Budapest, Hungary and Department of Mathematics, University College London, Gower Street, London, WC1E 6BT, UK.
E-mail: barany@renyi.hu

Roy Meshulam
Department of Mathematics, Technion - Israel Institute of Technology, Haifa 32000, Israel.
E-mail: meshulam@math.technion.ac.il

Eran Nevo
Einstein Institute of Mathematics, The Hebrew University of Jerusalem, Jerusalem 91904, Israel.
E-mail: nevo@math.huji.ac.il

Martin Tancer
Department of Applied Mathematics, Charles University in Prague, Malostranské náměstí 25, 118 00, Praha 1, Czech Republic.
E-mail: tancer@kam.mff.cuni.cz

$Z_1 \subset U_1, \ldots, Z_{d+1} \subset U_{d+1}$ *and a point* $u \in \mathbb{R}^d$ *such that each* $|Z_i| \geq c_1(d)n$ *and* $u \in \operatorname{conv}\{z_1, \ldots, z_{d+1}\}$ *for every choice of* $z_1 \in Z_1, \ldots, z_{d+1} \in Z_{d+1}$.

This result was proved by Bárány, Füredi, and Lovász [3] for $d = 2$ and by Pach [12] for general d. Here we show that this theorem does not admit a topological extension when the size of the Z_i is linear in n, but does admit one when the sizes are of order $(\log n)^{1/d}$. Now we reformulate Theorem 1.1 and then we state the topological extension.

Throughout the paper we will identify an abstract simplicial complex X with its geometric realization. For $k \geq 0$, let $X^{(k)}$ denote the k-dimensional skeleton of X and let $X(k)$ be the family of k-dimensional faces of X. For an abstract simplex $\sigma = \{v_0, \ldots, v_k\} \in X(k)$, we write $\langle v_0, \ldots, v_k \rangle$ for its geometric realization.

Let Δ_{n-1} denote the $(n-1)$-simplex. Consider $d+1$ sets $V_1, \ldots, V_{d+1}$, each of size n, and their join

$$(\Delta_{n-1}^{(0)})^{*(d+1)} \cong V_1 * \cdots * V_{d+1} := \{\sigma \subset \bigcup_{i=1}^{d+1} V_i : |\sigma \cap V_i| \leq 1 \text{ for all } 1 \leq i \leq d+1\}.$$

Trivially, there is an affine map $f : (\Delta_{n-1}^{(0)})^{*(d+1)} \to \mathbb{R}^d$ that is a bijection between V_i and U_i for each i (where U_i are the sets from the statement of Pach's theorem). In this setting the homogeneous selection theorem says that there exist subsets $Z_i \subset V_i$ such that $|Z_i| \geq c_1(d)n$ and

$$\bigcap_{z_1 \in Z_1, \ldots, z_{d+1} \in Z_{d+1}} f(\langle z_1, \ldots, z_{d+1} \rangle) \neq \emptyset.$$

Assume now that f is not affine but only continuous. For a mapping $f : (\Delta_{n-1}^{(0)})^{*(d+1)} \to \mathbb{R}^d$, let $\tau(f)$ denote the maximal m such that there exist m-element subsets $Z_1 \subset V_1, \ldots, Z_{d+1} \subset V_{d+1}$ that satisfy

$$\bigcap_{z_1 \in Z_1, \ldots, z_{d+1} \in Z_{d+1}} f(\langle z_1, \ldots, z_{d+1} \rangle) \neq \emptyset.$$

Define the *topological Pach number* $\tau(d,n)$ to be the minimum of $\tau(f)$ as f ranges over all continuous maps from $(\Delta_{n-1}^{(0)})^{*(d+1)}$ to $\mathbb{R}^d$. Our main result is the following:

Theorem 1.2. *For* $d \geq 1$ *there exists a constant* $c_2(d) = O(d)$ *such that* $\tau(d,n) \leq c_2(d) n^{1/d}$ *for all* $n \geq (2d)^d$.

For a lower bound on $\tau(d,n)$ we only have the following:

Theorem 1.3. *For* $d \geq 1$ *there exists a constant* $c_3(d) > 0$ *such that* $\tau(d,n) \geq c_3(d)(\log n)^{1/d}$ *for all* n.

Motivation and background. Theorem 1.1 is a descendant of the following selection theorem.

Theorem 1.4 (First selection theorem)**.** *Let P be a set of n-points in general position in $\mathbb{R}^d$. Then there is a point in at least $c_4(d)\binom{n}{d+1}$ d-simplices spanned by P.*

Theorem 1.4 was proved by Boros and Füredi [4] in the plane and it was generalized to arbitrary dimension by the first author [2]. Relatively recent extensive work of Gromov [9] implies a topological version of Theorem 1.4; see Theorem 4.1 for the precise statement of this extension. In addition, Gromov's approach yielded a significant improvement of the lower bound for the highest possible value of the constant $c_4(d)$ in Theorem 1.4.

From this point of view, it is desirable to know whether there is a topological extension of Theorem 1.1 which could also possibly be quantitatively stronger with respect to the constant $c_1(d)$. However, Theorem 1.2 shows that in the case of this homogeneous selection theorem we would ask for too much.

A brief proof overview. Our proof of Theorem 1.2 partially builds on the approach from [14] where the homogeneous selection theorem was used to distinguish a geometric and a topological invariant.

For the proof of Theorem 1.2 we need to exhibit a continuous map $f\colon (\Delta_{n-1}^{(0)})^{*(d+1)} \to \mathbb{R}^d$ such that $\tau(f)$ is low, namely at most $c_2(d)n^{1/d}$. Our result is in fact stronger: For some $N \geq (d+1)n$, we construct a map $f\colon \Delta_{N-1} \to \mathbb{R}^d$ such that for *any* pairwise disjoint n-subsets $V_1,\dots,V_{d+1}$ of the vertex set of Δ_{N-1}, the restriction of f to $V_1 * \cdots * V_{d+1} \cong (\Delta_{n-1}^{(0)})^{*(d+1)}$ satisfies

$$\tau(f_{|V_1*\cdots*V_{d+1}}) \leq c_2(d)n^{1/d}. \tag{1}$$

The construction of f proceeds roughly as follows (see Sections 2 and 3 for the relevant definitions). Let L be any finite graded lattice of rank $d+1$ with minimal element $\widehat{0}$, whose set of atoms A satisfies $|A| = N \geq n(d+1)$. Let $S(A) \cong \Delta_{N-1}$ be the simplex on the vertex set A, and let $\tilde{L} = L - \{\widehat{0}\}$. We first observe (see Claim 3.2) that there exists a continuous map g from $S(A)$ to the order complex $\Delta(\tilde{L})$ such that $g(\langle a_0,\dots,a_p\rangle) \subset \Delta(\tilde{L}_{\leq \bigvee_{i=0}^{p} a_i})$ for any atoms $a_0,\dots,a_p \in A$ (in words: $\langle a_0,\dots,a_p\rangle$ maps into the subcomplex below the join of the atoms $a_0,\dots,a_p \in A$ in the order complex of $\tilde{L}$). Next we define $f : S(A) \to \mathbb{R}^d$ as the composition $e \circ g$, where $e : \Delta(\tilde{L}) \to \mathbb{R}^d$ is the affine extension of a generic map from $\tilde{L}$ to $\mathbb{R}^d$.

Our main technical result, Theorem 2.1, provides an upper bound on $\tau(f_{|V_1*\cdots*V_{d+1}})$ in terms of the expansion of the bipartite graph G_L of atoms vs. coatoms of L. The desired bound (1) follows from Theorem 2.1 by choosing L to be the lattice of linear subspaces of the vector space $\mathbb{F}_q^{d+1}$ over the finite field with q elements (for suitable $q = q(n,d)$), and utilizing a well known expansion property of the corresponding graph G_L.

The paper is organized as follows: In Section 2 we state Theorem 2.1 and apply it to prove Theorem 1.2. The proof of Theorem 2.1 is given in Section 3. In Section 4 we prove Theorem 1.3 as a direct application of results of Gromov [9] and Erdős [8].

Subsequent work. Considering our work, Bukh and Hubard [5] very recently improved the bound on $\tau(d,n)$ to $\tau(d,n) \leq 30(\ln n)^{1/(d-1)}$.

2 Finite Lattices and Topological Pach Numbers

A finite poset $(L,<)$ is a *lattice* if for any two element $x,y \in L$ the set $\{z : z \leq x, z \leq y\}$ has a unique maximal element $x \wedge y$, and the set $\{z : z \geq x, z \geq y\}$ has a unique minimal element $x \vee y$. In particular, a lattice has a minimal element $\widehat{0}$ and a maximal element $\widehat{1}$. A lattice L is *graded* with rank function $\mathrm{rk} : L \to \mathbb{N}$, if $\mathrm{rk}(\widehat{0}) = 0$ and if $\mathrm{rk}(y) = \mathrm{rk}(x) + 1$ whenever y covers x (i.e. $\{z : x \leq z \leq y\} = \{x,y\}$). See Stanley's book [13] for a comprehensive reference on the combinatorics of posets and lattices.

Let L be a graded lattice of rank $\mathrm{rk}(\widehat{1}) = d+1$. Let

$$A = \{x \in L : \mathrm{rk}(x) = 1\} \ \ , \ \ C = \{x \in L : \mathrm{rk}(x) = d\}$$

be respectively the sets of *atoms* and *coatoms* of L. For $x \in L$ let

$$A_x = \{a \in A : a \leq x\} \ \ , \ \ C_x = \{c \in C : x \leq c\}.$$

Let G_L denote the bipartite graph on the vertex set $A \cup C$ with edges $(a,c) \in A \times C$ iff $a \leq c$. For a set of atoms $Z \subset A$ let $\Gamma(Z) = \cup_{z \in Z} C_z$ be the neighborhood of Z.

The main ingredient of the proof of Theorem 1.2 is the following connection between $\tau(d,n)$ and the expansion of G_L.

Theorem 2.1. *Let L be a graded lattice of rank $d+1$ such that $|A| \geq n(d+1)$. Then $m = \tau(d,n)$ satisfies*

$$\min_{Z \subset A, |Z|=m} |\Gamma(Z)| \leq \frac{d}{d+1}\big(\max_{a \in A} |C_a| + |C|\big).$$

The proof of Theorem 2.1 is deferred to Section 3.

Proof of Theorem 1.2: Let $n \geq (2d)^d$. By Bertrand's postulate there exists a prime q such that

$$2d \leq \big((d+1)n\big)^{1/d} \leq q \leq 2\big((d+1)n\big)^{1/d}. \tag{2}$$

Let $\mathbb{F}_q$ be the finite field of order q. Let $L = L(d+1,q)$ denote the graded lattice of linear subspaces of $\mathbb{F}_q^{d+1}$ ordered by inclusion, with the natural rank function $\mathrm{rk}(x) = \dim x$ for all $x \in L$. The sets of atoms and coatoms of L satisfy $|A| = |C| = N_d = \frac{q^{d+1}-1}{q-1}$ and $|C_a| = N_{d-1} = \frac{q^d-1}{q-1}$ for all $a \in A$. Any two distinct 1-dimensional subspaces of $\mathbb{F}_q^{d+1}$ are contained in exactly $N_{d-2} = \frac{q^{d-1}-1}{q-1}$ hyperplanes of $\mathbb{F}_q^{d+1}$. Hence, if $a \neq a' \in A$ are two distinct atoms then

$$|C_a \cap C_{a'}| = N_{d-2} = \frac{q^{d-1}-1}{q-1}.$$

It follows that if $Z \subset A$, then the family $\{C_a : a \in Z\}$ forms an N_{d-1}-uniform hypergraph on vertex set $\Gamma(Z)$ with $|Z|$ edges, and any two distinct edges intersect in a set of size N_{d-2}. Applying a result of Corrádi [6] (see also exercise 13.13 in [10] and Theorem 2.3(ii) in [1]) we obtain the following lower bound on the expansion of G_L.

$$\begin{aligned}|\Gamma(Z)| &\geq \frac{|Z|N_{d-1}^2}{N_{d-1} + (|Z|-1)N_{d-2}} = \frac{|Z|N_{d-1}^2}{q^{d-1} + |Z|N_{d-2}} \\ &= N_d - \frac{q^{d-1}(N_d - |Z|)}{q^{d-1} + |Z|N_{d-2}} \geq N_d - \frac{q^{d-1}N_d}{|Z|N_{d-2}} \\ &\geq N_d - \frac{qN_d}{|Z|} \geq N_d - \frac{N_d^{1+\frac{1}{d}}}{|Z|}.\end{aligned} \tag{3}$$

Next note that (2) implies that $|A| = N_d \geq q^d \geq (d+1)n$. Applying Theorem 2.1 together with (3), it follows that $m = \tau(d,n)$ satisfies

$$\begin{aligned}N_d - \frac{N_d^{1+\frac{1}{d}}}{m} &\leq \min_{Z \subset A, |Z|=m} |\Gamma(Z)| \\ &\leq \frac{d}{d+1}\big(\max_{a\in A}|C_a| + |C|\big) \\ &= \frac{d}{d+1}(N_{d-1} + N_d).\end{aligned} \tag{4}$$

The assumption $q \geq 2d$ implies that

$$\begin{aligned}\frac{N_d}{N_d - dN_{d-1}} &= \frac{q^{d+1}-1}{q^{d+1} - 1 - d(q^d - 1)} \\ &\leq \frac{q^{d+1}}{q^{d+1} - dq^d} = \frac{q}{q-d} \leq 2.\end{aligned} \tag{5}$$

Rearranging (4) and using (5) and $q^d \leq 2^d(d+1)n$, we obtain

$$\begin{aligned}m &\leq \frac{(d+1)N_d^{1+\frac{1}{d}}}{N_d - dN_{d-1}} \leq 2(d+1)N_d^{\frac{1}{d}} \\ &\leq 2(d+1)\big((d+1)q^d\big)^{1/d} \\ &\leq 2(d+1)\big((d+1)(2^d(d+1)n)\big)^{1/d} \\ &= 4(d+1)\big((d+1)^2 n\big)^{1/d}.\end{aligned}$$

□

3 Continuous Maps of Finite Lattices

In this section we prove Theorem 2.1. We first recall some definitions. The *order complex* $\Delta(P)$ of a finite poset $(P,<)$ is the simplicial complex on the vertex set P, whose k-simplices are the chains $x_0 < \cdots < x_k$ in P.

Let L be a graded lattice of rank $d+1$ and let $\tilde{L} = L - \{\widehat{0}\}$. For a subset $\sigma \subset L$ let $\vee\sigma = \vee_{x\in\sigma} x$. Let $S(A)$ be the simplex on the set A of atoms of L (identified as usual with its geometric realization). For $x \in \tilde{L}$ let $\tilde{L}_{\leq x} = \{y \in \tilde{L} : y \leq x\}$. The main ingredient in the proof of Theorem 2.1 is the following result.

Proposition 3.1. *There exists a continuous map $f : S(A) \to \mathbb{R}^d$ such that for any $u \in \mathbb{R}^d$*

$$|\{c \in C : u \in f(\langle A_c \rangle)\}| \leq d \max_{a\in A} |C_a|. \tag{6}$$

(Note that, in accordance with our notation, $\langle A_c \rangle$ stands here for the geometric realization of A_c, considered as a face of $S(A)$.)

We first note the following

Claim 3.2. *There exists a continuous map $g : S(A) \to \Delta(\tilde{L})$ such that for all $x \in \tilde{L}$*

$$g(\langle A_x \rangle) \subset \Delta(\tilde{L}_{\leq x}).$$

Proof: We define g inductively on the k-skeleton $S(A)^{(k)}$. On the vertices $a \in A$ of $S(A)$ let $g(a) = a$. Let $0 < k \leq |A|-1$ and suppose g has been defined on $S(A)^{(k-1)}$. Let $\sigma = \langle a_0, \ldots, a_k \rangle \in S(A)^{(k)}$ and let $y = \vee\sigma$. For $0 \leq i \leq k$ let

$$\sigma_i = \langle a_0, \ldots, a_{i-1}, \widehat{a_i}, a_{i+1}, \ldots, a_k \rangle$$

be the i-th face of σ. Let $y_i = \vee\sigma_i$. Then g is defined on σ_i and by induction hypothesis

$$g(\sigma_i) \subset \Delta(\tilde{L}_{\leq y_i}) \subset \Delta(\tilde{L}_{\leq y}).$$

Being a cone, $\Delta(\tilde{L}_{\leq y})$ is contractible and hence g can be continuously extended from the boundary $\partial\sigma$ to the whole of σ so that $g(\sigma) \subset \Delta(\tilde{L}_{\leq y})$. It follows in particular that for $x \in \tilde{L}$

$$g(\langle A_x \rangle) \subset \Delta(\tilde{L}_{\leq \vee A_x}) \subset \Delta(\tilde{L}_{\leq x}).$$

□

Proof of Proposition 3.1: By a general position argument we choose a mapping $e : \tilde{L} \to \mathbb{R}^d$ with the following property: For any pairwise disjoint subsets $S_1, \ldots, S_{d+1} \subset \tilde{L}$ of cardinalities $|S_i| \leq d$, it holds that

$$\bigcap_{i=1}^{d+1} \operatorname{aff}\big(e(S_i)\big) = \emptyset,$$

and thus in particular

$$\bigcap_{i=1}^{d+1} \operatorname{relint}\operatorname{conv}\big(e(S_i)\big) = \emptyset. \tag{7}$$

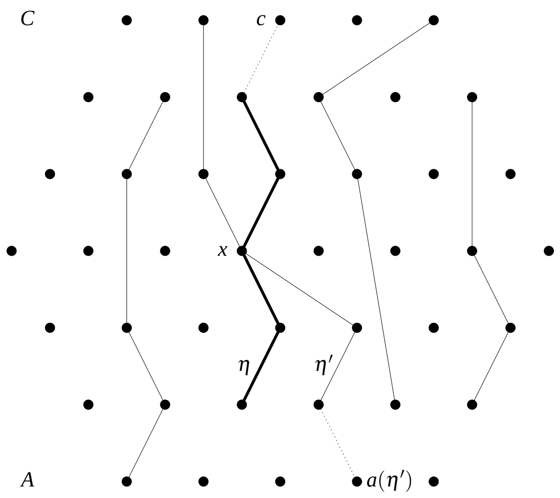

Fig. 1: The bold chain corresponds to η. The other chains represent simplices of T'.

Extend e by linearity to the whole of $\Delta(\tilde{L})$ and let $f = e \circ g : S(A) \to \mathbb{R}^d$, where g is the map from Claim 3.2. We claim that the map f satisfies (6). Let $u \in \mathbb{R}^d$ and let

$$T = \{\eta \in \Delta(\tilde{L}) : u \in \operatorname{relint} e(\langle \eta \rangle)\}.$$

Choose a maximal pairwise disjoint subfamily $T' \subset T$. It follows by (7) that $|T'| \leq d$. For each $\eta' \in T'$ choose an atom $a(\eta') \in A$ such that

$$a(\eta') \leq \min \eta'. \tag{8}$$

Now let $c \in C$ be such that $u \in f(\langle A_c \rangle)$. Then there exists a $b \in g(\langle A_c \rangle) \subset \Delta(\tilde{L}_{\leq c})$ such that $u = e(b)$. Let $\eta \in T$ be such that $b \in \operatorname{relint}\langle \eta \rangle$. Then

$$\eta \in \Delta(\tilde{L}_{\leq c}). \tag{9}$$

By maximality of T' there exists a simplex $\eta' \in T'$ and a vertex $x \in \eta' \cap \eta$. It follows by (8) and (9) that $a(\eta') \leq x \leq c$, i.e. $c \in C_{a(\eta')}$ (see figure 1). Therefore

$$|\{c \in C : u \in f(\langle A_c \rangle)\}| \leq \sum_{\eta' \in T'} |C_{a(\eta')}| \leq d \max_{a \in A} |C_a|.$$

□

Proof of Theorem 2.1: Let L be a lattice of rank $d+1$ whose set of atoms A satisfies $|A| \geq (d+1)n$. Let $V_1, \ldots, V_{d+1}$ be disjoint n-subsets of A. By Proposition 3.1 there exists a continuous map $f : S(A) \to \mathbb{R}^d$ such that for any $u \in \mathbb{R}^d$

$$|\{c \in C : u \in f(\langle A_c \rangle)\}| \leq d \max_{a \in A} |C_a|.$$

Let $m = \tau(d,n)$. Then there exist $Z_1 \subset V_1, \ldots, Z_{d+1} \subset V_{d+1}$ and a $u \in \mathbb{R}^d$ such that $|Z_i| \geq m$ for all $1 \leq i \leq d+1$ and

$$u \in \bigcap_{z_1 \in Z_1, \ldots, z_{d+1} \in Z_{d+1}} f(\langle z_1, \ldots, z_{d+1} \rangle).$$

Write

$$C(Z_1, \ldots, Z_{d+1}) = \bigcap_{i=1}^{d+1} \{c \in C : A_c \cap Z_i \neq \emptyset\}.$$

If $c \in C(Z_1, \ldots, Z_{d+1})$ then there exist $z_1 \in Z_1, \ldots, z_{d+1} \in Z_{d+1}$ such that $z_i \leq c$ for all i and hence $u \in f(\langle z_1, \ldots, z_{d+1} \rangle) \subset f(\langle A_c \rangle)$. Hence by Proposition 3.1

$$|C(Z_1, \ldots, Z_{d+1})| \leq d \max_{a \in A} |C_a|. \tag{10}$$

On the other hand

$$\begin{aligned} |C(Z_1, \ldots, Z_{d+1})| &= |C - \bigcup_{i=1}^{d+1} (C - \Gamma(Z_i))| \\ &\geq |C| - \sum_{i=1}^{d+1} (|C| - |\Gamma(Z_i)|) = \sum_{i=1}^{d+1} |\Gamma(Z_i)| - d|C| \\ &\geq (d+1) \min_{Z \subset A, |Z|=m} |\Gamma(Z)| - d|C|. \end{aligned} \tag{11}$$

Theorem 2.1 now follows from (10) and (11).

□

Remark: The mapping $g : S(A) \to \Delta(\tilde{L})$ constructed in Claim 3.2 is in general not simplicial. It follows (as of course must be the case by Theorem 1.1) that $f = e \circ g : S(A) \to \mathbb{R}^d$ is not affine.

4 The Lower Bound

Theorem 1.3 is a direct consequence of Gromov's topological overlap Theorem [9] combined with a result of Erdős on complete $(d+1)$-partite subhypergraphs in $(d+1)$-uniform dense hypergraphs [8]. We first recall these results. Let X be a finite d-dimensional pure simplicial complex. For $k \geq 0$, let $f_k(X) = |X(k)|$ denote the number of k-dimensional faces of X. Define a positive weight function $w = w_X$ on the simplices of X as follows. For $\sigma \in X(k)$, let $c(\sigma) = |\{\eta \in X(d) : \sigma \subset \eta\}|$ and let

$$w(\sigma) = \frac{c(\sigma)}{\binom{d+1}{k+1} f_d(X)}.$$

Let $C^k(X)$ denote the space of $\mathbb{F}_2$-valued k-cochains of X with the coboundary map $d_k : C^k(X) \to C^{k+1}(X)$. As usual, the space of k-coboundaries is denoted by $d_{k-1}\big(C^{k-1}(X)\big) = B^k(X)$. For $\phi \in C^k(X)$, let $[\phi]$ denote the image of ϕ in $C^k(X)/B^k(X)$. Let

$$\|\phi\| = \sum_{\sigma \in X(k) : \phi(\sigma) \neq 0} w(\sigma)$$

and

$$\|[\phi]\| = \min\{\|\phi + d_{k-1}\psi\| : \psi \in C^{k-1}(X)\}.$$

The *k-th coboundary expansion constant* of X is

$$h_k(X) = \min\left\{\frac{\|d_k\phi\|}{\|[\phi]\|} : \phi \in C^k(X) - B^k(X)\right\}.$$

Note that $h_k(X) = 0$ iff $\tilde{H}^k(X;\mathbb{F}_2) \neq 0$. One may regard $h_k(X)$ as a sort of distance between X and the family of complexes Y that satisfy $\tilde{H}^k(Y;\mathbb{F}_2) \neq 0$. Gromov's celebrated topological overlap result is the following:

Theorem 4.1 (Gromov [9]). *For any integer $d \geq 0$ and any $\varepsilon > 0$ there exists a $\delta = \delta(d,\varepsilon) > 0$ such that if $h_k(X) \geq \varepsilon$ for all $0 \leq k \leq d-1$, then for any continuous map $f : X \to \mathbb{R}^d$ there exists a point $u \in \mathbb{R}^d$ such that*

$$|\{\sigma \in X(d) : u \in f(\sigma)\}| \geq \delta f_d(X).$$

We next describe a result of Erdős that generalizes the well known Erdős-Stone and Kővári-Sós-Turán theorems from graphs to hypergraphs.

Theorem 4.2 (Erdős [8]). *For any d and $c' > 0$ there exists a constant $c = c(d,c') > 0$ such that for any $(d+1)$-uniform hypergraph $\mathscr{F}$ on N-element set V with at least $c'N^{d+1}$ hyperedges, there exists an $m \geq c(\log N)^{1/d}$ and disjoint m-element sets $Z_1,\ldots,Z_{d+1} \subset V$ such that $\{z_1,\ldots,z_{d+1}\} \in \mathscr{F}$ for all $z_1 \in Z_1,\ldots,z_{d+1} \in Z_{d+1}$.*

Proof of Theorem 1.3: Recall that $V_1,\ldots,V_{d+1}$ are disjoint n-element sets and let $V = V_1 \cup \cdots \cup V_{d+1}$, $|V| = N = (d+1)n$. Let $X = V_1 * \ldots * V_{d+1}$ and let $f : X \to \mathbb{R}^d$ be a continuous map. It was shown by Gromov [9] (see also [7,11]) that the expansion constants $h_i(X)$ are uniformly bounded away from zero. Concretely, it follows from Theorem 3.3 in [11] that $h_i(X) \geq \varepsilon = 2^{-d}$ for $0 \leq i \leq d-1$. Let $\delta = \delta(d, 2^{-d})$. Then by Theorem 4.1 there exists a $u \in \mathbb{R}^d$ and a family $\mathscr{F} \subset X(d)$ of cardinality

$$|\mathscr{F}| \geq \delta f_d(X) = \delta n^{d+1} = \delta(d+1)^{-(d+1)}N^{d+1}$$

such that $u \in f(\sigma)$ for all $\sigma \in \mathscr{F}$. Writing $c' = \delta(d+1)^{-(d+1)}$ and $c_3(d) = c(d,c')$, it follows from Theorem 4.2 that there exists an $m \geq c_3(d)(\log N)^{1/d} \geq c_3(d)(\log n)^{1/d}$ and disjoint m-sets $Z_1,\cdots,,Z_{d+1} \subset V$ such that $u \in f(\langle z_1,\ldots,z_{d+1}\rangle)$ for all $z_1 \in Z_1,\ldots,z_{d+1} \in Z_{d+1}$. Clearly, there exists a permutation π on $\{1,\ldots,d+1\}$ such that $Z_{\pi(i)} \subset V_i$ for all $1 \leq i \leq d+1$.

$\square$

Acknowledgements This research was supported by ERC Advanced Research Grant no 267165 (DISCONV). Imre Bárány is partially supported by Hungarian National Research Grant K 111827. Roy Meshulam is partially supported by ISF grant 326/16 and GIF grant 1261/14, Eran Nevo by ISF grant 1695/15 and Martin Tancer by GAČR grant 16-01602Y.

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