On the Homological Dimension of Lattices

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Abstract

Let L be a finite lattice and let $\widehat{L} = L - \{\hat{0}, \hat{1}\}$. It is shown that if the order complex $\Delta(\widehat{L})$ satisfies $\tilde{\mathrm{H}}_{k-2}\big(\Delta(\widehat{L})\big) \neq 0$ then $|L| \geq 2^k$. Equality $|L| = 2^k$ holds iff L is isomorphic to the Boolean lattice $\{0,1\}^k$.

1 Introduction

The order complex $\Delta(P)$ of a finite poset $(P, \leq)$ is the simplicial complex on the vertex set P whose simplices are the increasing chains $x_0 < \cdots < x_k$ (see [1] for a thorough discussion). A result of Kozlov ([2]) implies that if the integral homology group $\tilde{\mathrm{H}}_k(\Delta(P))$ is non-trivial then $|P| \geq 2(k+1)$, with equality $|P| = 2(k+1)$ iff P is isomorphic to the ordinal sum of $k+1$ antichains of size 2. In this note we consider a similar question for lattices. For a lattice L with minimal element $\hat{0}$ and maximal element $\hat{1}$, let $\widehat{L} = L - \{\hat{0}, \hat{1}\}$.

Example: Let B_k denote the Boolean lattice $\{0,1\}^k$. The order complex $\Delta(\widehat{B_k})$ coincides with the barycentric subdivision of the boundary of the $(k-1)$-simplex, hence $\tilde{\mathrm{H}}_{k-2}\big(\Delta(\widehat{B_k})\big) \neq 0$. Here we show that B_k is the unique smallest lattice with non-trivial $(k-2)$-dimensional homology.

Theorem 1.1. *If a lattice L satisfies $\tilde{\mathrm{H}}_{k-2}\big(\Delta(\widehat{L})\big) \neq 0$, then $|L| \geq 2^k$. Equality $|L| = 2^k$ holds iff $L \cong B_k$.*

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Let $\tilde{\beta}_i(\Delta(\widehat{L}))$ denote the i-th reduced Betti number of $\Delta(\widehat{L})$. Then $\mu(L)$, the Möbius function of L, is equal to the reduced Euler characteristic $\tilde{\chi}(\Delta(\widehat{L})) = \sum_i (-1)^i \tilde{\beta}_i(\Delta(\widehat{L}))$. It is conjectured that if $|L| = n$ then $\mu(L) = o(n^2)$. Edelman and Kahn have shown (see [5]) that $\mu(L) \leq n^{log_2 n}$. Theorem 1.1 implies the following slightly stronger result.

Corollary 1.2. *If L is a lattice on n elements then*

$$\sum_i \tilde{\beta}_i(\Delta(\widehat{L})) \leq n^{\log_2 n} .$$

In Section 2 we establish Theorem 1.1 and Corollary 1.2 using an idea from the Edelman-Kahn proof of the upper bound on $\mu(L)$.

We conclude the introduction with a conjecture concerning the relation between the reduced Betti numbers of $\Delta(\widehat{L})$ and the cardinality of L. For a prime power q, let $L_k(q)$ denote the lattice of linear subspaces of the k-dimensional vector space $\mathbb{F}_q^k$ over the field $\mathbb{F}_q$. Then $\tilde{\beta}_{k-2}(\Delta(\widehat{L_k(q)})) = q^{\binom{k}{2}}$ and $|L_k(q)| = \sum_{i=0}^{k} \begin{bmatrix} k \\ i \end{bmatrix}_q$ where $\begin{bmatrix} k \\ i \end{bmatrix}_q = \prod_{j=0}^{i-1} \frac{q^k - q^j}{q^i - q^j}$ is the number of i-dimensional subspaces of $\mathbb{F}_q^k$ (see e.g. [4]).

Conjecture 1.3. *If a lattice L satisfies $\tilde{\beta}_{k-2}(\Delta(\widehat{L})) \geq q^{\binom{k}{2}}$ then $|L| \geq \sum_{i=0}^{k} \begin{bmatrix} k \\ i \end{bmatrix}_q$.*

Note that $\lim_{q \to 1} \begin{bmatrix} k \\ i \end{bmatrix}_q = \binom{k}{i}$. Hence Theorem 1.1 may be viewed as the $q = 1$ case of Conjecture 1.3.

2 Homological Dimension of Lattices

We prove Theorem 1.1 by induction on k. The case $k = 1$ is clear. Suppose $k \geq 2$ and let L be a lattice of minimal cardinality such that $\tilde{\mathrm{H}}_{k-2}(\Delta(\widehat{L})) \neq 0$. Let x be an arbitrary atom of L. The following observation is due to Quillen ([3]). For completeness we include the proof.

Claim 2.1. *If $x \leq y$ for all coatoms $y \in L$, then $\Delta(\widehat{L})$ is contractible.*

Proof: Let P, Q be two posets. Recall (see [1]) that if $f, g : P \to Q$ are two order-preserving maps such that $f(p) \leq g(p)$ for all $p \in P$, then the induced simplicial maps $f, g : \Delta(P) \to \Delta(Q)$ are homotopic. Now let $P = Q = \widehat{L}$ and for $p \in P$ let $f(p) = x$, $g(p) = p \vee x$ and $h(p) = p$. Then $f(p) \leq g(p) \geq h(p)$ for all p and thus

$$\text{constant map } = f \simeq g \simeq h = \text{ identity map .}$$

Therefore $\Delta(\widehat{L})$ is contractible.

□

By Claim 2.1 there exists a coatom y such that $x \not\leq y$. Consider the lattices $L' = [\hat{0}, y]$ and $L'' = [x, \hat{1}]$. Then $L' \cap L'' = \emptyset$, hence either $|L'| \leq \frac{|L|}{2}$ or $|L''| \leq \frac{|L|}{2}$. By reversing the order if necessary, we may assume that $|L''| \leq \frac{|L|}{2}$. Let M be the lattice $L - \{x\}$. Write $A = \Delta(\widehat{L})$, $B = \Delta\big([x, \hat{1})\big)$, $C = \Delta(\widehat{M})$ and $D = \Delta(\widehat{L''})$. Then $A = B \cup C$ and $B \cap C = D$. Note that $\tilde{\mathrm{H}}_{k-2}(B) = 0$ since B is a cone on x, and $\tilde{\mathrm{H}}_{k-2}(C) = 0$ by the minimality assumption on L. It thus follows from the Mayer-Vietoris sequence

$$0 = \tilde{\mathrm{H}}_{k-2}(B) \oplus \tilde{\mathrm{H}}_{k-2}(C) \to \tilde{\mathrm{H}}_{k-2}(A) \to \tilde{\mathrm{H}}_{k-3}(D)$$

that $\tilde{\mathrm{H}}_{k-3}(D) \neq 0$. Therefore by induction $|[x, \hat{1}]| = |L''| \geq 2^{k-1}$, hence $|L| \geq 2|L''| \geq 2^k$.

Suppose now that $|L| = 2^k$. The argument above implies that $|L'| = |L''| = 2^{k-1}$ and that both $\tilde{\mathrm{H}}_{k-3}\big(\Delta(\widehat{L'})\big)$ and $\tilde{\mathrm{H}}_{k-3}\big(\Delta(\widehat{L''})\big)$ are non-trivial. Therefore by the induction hypothesis $L' \cong B_{k-1} \cong L''$.

Let $x_1, \ldots, x_m$ denote the atoms of L, and for each $1 \leq i \leq m$ let y_i be a coatom of L such that $x_i \not\leq y_i$. Let $L'_i = [\hat{0}, y_i]$ and $L''_i = [x_i, \hat{1}]$. By the above it follows that $L'_i \cup L''_i = L$ and $L'_i \cong B_{k-1} \cong L''_i$. Note that the first statement implies that $x_i \leq y_j$ for $i \neq j$.

For a vector $u = (u_1, \ldots, u_{k-1}) \in \{0,1\}^{k-1} = B_{k-1}$ and an $\epsilon \in \{0,1\}$ let $S_\epsilon(u) = \{1 \leq i \leq k-1 : u_i = \epsilon\}$. Since $x_1, \ldots, x_{m-1}$ are the atoms of $L'_m \cong B_{k-1}$, it follows that $m = k$ and that the map $\phi_0 : B_{k-1} \to L'_k$ given by $\phi_0(u) = \bigvee_{i \in S_1(u)} x_i$ is an isomorphism. Similarly, the map $\phi_1 : B_{k-1} \to L''_k$ given by $\phi_1(u) = \bigwedge_{i \in S_0(u)} y_i$ is an isomorphism. We now claim that the map $\phi : B_k \to L$ given on $(u, \epsilon) \in \{0,1\}^{k-1} \times \{0,1\}$ by $\phi(u, \epsilon) = \phi_\epsilon(u)$ is an isomorphism. ϕ is clearly bijective. It remains to show that ϕ is a lattice map. We will check that ϕ preserves the meet, i.e. $\phi_{\epsilon \wedge \epsilon'}(u \wedge v) = \phi_\epsilon(u) \wedge \phi_{\epsilon'}(v)$.

If $\epsilon = \epsilon'$ then this follows from the corresponding property of ϕ_ϵ . For $\epsilon = 0, \epsilon' = 1$ we have to show that

$$\bigvee_{i \in S_1(u \wedge v)} x_i = \bigvee_{i \in S_1(u)} x_i \ \wedge \bigwedge_{j \in S_0(v)} y_j \ . \tag{1}$$

The righthand side of (1) belongs to L'_m hence

$$\bigvee_{i \in S_1(u)} x_i \ \wedge \bigwedge_{j \in S_0(v)} y_j = \bigvee_{i \in S_1(w)} x_i$$

for some $w \in \{0,1\}^{k-1}$. Since $x_i \not\leq y_i$ for all i it follows that $w \leq u \wedge v$. The reverse inequality $u \wedge v \leq w$ is a consequence of the fact that $x_i \leq y_j$ for all $i \neq j$. Therefore $w = u \wedge v$ and thus (1) holds. The proof that ϕ preserves the join is similar.

□

Proof of Corollary 1.2: Let $|L| = n$. Since $\Delta(\widehat{L})$ is a complex on $n-2$ vertices its Betti numbers satisfy $\tilde{\beta}_i\big(\Delta(\widehat{L})\big) \leq \binom{n-3}{i+1}$ for all i. Theorem 1.1 on the other hand implies that $\tilde{\beta}_i\big(\Delta(\widehat{L})\big) = 0$ for $i > \log_2 n - 2$. Hence

$$\sum_i \tilde{\beta}_i\big(\Delta(\widehat{L})\big) \leq \sum_{i \leq \log_2 n - 2} \binom{n-3}{i+1} < n^{log_2 n} \ .$$

□

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