Homological Connectivity of Random k-dimensional Complexes

R. Meshulam[*] N. Wallach[†]

September 27, 2006

Abstract

Let Δ_{n-1} denote the $(n-1)$-dimensional simplex. Let Y be a random k-dimensional subcomplex of Δ_{n-1} obtained by starting with the full $(k-1)$-dimensional skeleton of Δ_{n-1} and then adding each k-simplex independently with probability p. Let $H_{k-1}(Y;R)$ denote the $(k-1)$-dimensional reduced homology group of Y with coefficients in a finite abelian group R. It is shown that for any fixed R and $k \geq 1$ and for any function $\omega(n)$ that tends to infinity

$$\lim_{n\to\infty} \Pr\left[\ H_{k-1}(Y;R)=0\ \right] = \begin{cases} 0 & p=\frac{k\log n-\omega(n)}{n} \\ 1 & p=\frac{k\log n+\omega(n)}{n} \end{cases}$$

1 Introduction

Let $G(n,p)$ denote the probability space of graphs on the vertex set $[n]=\{1,\ldots,n\}$ with independent edge probabilities p. Let log denote the natural logarithm. A classical result of Erdős and Rényi [2] asserts that the threshold probability for connectivity of $G \in G(n,p)$ coincides with the threshold for

[*]Institute for Advanced Study, Princeton, NJ 08540 and Department of Mathematics, Technion, Haifa 32000, Israel. Supported by a state of New Jersey grant and by the Israel Science Foundation. e-mail: meshulam@math.technion.ac.il

[†]Department of Mathematics, Technion, Haifa 32000, Israel. e-mail: tani@tx.technion.ac.il

the non-existence of isolated vertices in G . In particular, for any function $\omega(n)$ that tends to infinity

$$\lim_{n\to\infty} \Pr [\; G \in G(n,p) \; : \; G \; \text{ connected }] = \begin{cases} 0 & p = \frac{\log n - \omega(n)}{n} \\ 1 & p = \frac{\log n + \omega(n)}{n} \end{cases} .$$

A 2-dimensional analogue of the Erdős-Rényi result was considered in [3], where the threshold for homological 1-connectivity of random 2-dimensional complexes was determined (see below). In this paper we study the homological $(k-1)$-connectivity of random k-dimensional complexes for a general fixed k.

We recall some topological terminology (see e.g. [4]) . Let X be a finite simplicial complex on the vertex set V. let $X^{(k)} = \{\sigma \in X : \dim \sigma \leq k\}$ denote the k-dimensional skeleton of X, and let $X(k)$ denote the set of k-dimensional simplices in X, each taken with an arbitrary but fixed orientation. Denote by $f_k(X) = |X(k)|$ the number of k-dimensional simplices in X. Let R be a fixed finite abelian group of cardinality r. A simplicial k-cochain is an R-valued skew-symmetric function on all ordered k-simplices of X. For $k \geq 0$ let $C^k(X)$ denote the group of k-cochains on X. The i-face of an ordered $(k+1)$-simplex $\sigma = [v_0, \ldots, v_{k+1}]$ is the ordered k-simplex $\sigma_i = [v_0, \ldots, \widehat{v_i}, \ldots, v_{k+1}]$. The coboundary operator $d_k : C^k(X) \to C^{k+1}(X)$ is given by

$$d_k\phi(\sigma) = \sum_{i=0}^{k+1} (-1)^i \phi(\sigma_i) \quad .$$

It is convenient to augment the cochain complex $\{C^i(X)\}_{i=0}^{\infty}$ with the (-1)-degree term $C^{-1}(X) = R$ with the coboundary map $d_{-1} : C^{-1}(X) \to C^0(X)$ given by $d_{-1}a(v) = a$ for $a \in R$, $v \in V$. Let $Z^k(X) = \ker(d_k)$ denote the space of k-cocycles and let $B^k(X) = \operatorname{Im}(d_{k-1})$ denote the space of k-coboundaries. For $k \geq 0$ let $H^k(X;R) = Z^k(X)/B^k(X)$ denote the k-th reduced cohomology group of X with coefficients in R. We abbreviate $H^k(X) = H^k(X;R)$.

Let Δ_{n-1} denote the $(n-1)$-dimensional simplex on the vertex set $V = [n]$. Let $Y_k(n,p)$ denote the probability space of complexes $\Delta_{n-1}^{(k-1)} \subset Y \subset \Delta_{n-1}^{(k)}$ with probability measure

$$\Pr(Y) = p^{f_k(Y)}(1-p)^{\binom{n}{k+1} - f_k(Y)} \; .$$

A $(k-1)$-simplex $\sigma \in \Delta_{n-1}(k-1)$ is *isolated* in Y if it is not contained in any of the k-simplices of Y. If σ is isolated then the indicator function of σ is a non-trivial $(k-1)$-cocycle of Y, hence $H^{k-1}(Y) \neq 0$. Our main result is that the threshold probability for the vanishing of $H^{k-1}(Y)$ coincides with the threshold for the non-existence of isolated $(k-1)$-simplices in Y.

Theorem 1.1. *Let $k \geq 1$ and R be fixed, and let $\omega(n)$ be any function which satisfies $\omega(n) \to \infty$ then*

$$\lim_{n\to\infty} \Pr \left[\ Y \in Y_k(n,p) : H^{k-1}(Y;R) = 0 \ \right] = \begin{cases} 0 & p = \frac{k \log n - \omega(n)}{n} \\ 1 & p = \frac{k \log n + \omega(n)}{n} \end{cases} \qquad (1)$$

Remarks:
1. Theorem 1.1 remains true when $H^{k-1}(Y)$ is replaced by the $(k-1)$-th reduced homology group $H_{k-1}(Y) = H_{k-1}(Y;R)$. This follows from the universal coefficient theorem since $H_{k-2}(Y) = 0$ for $Y \in Y_k(n,p)$.
2. The $k=1$ case of Theorem 1.1 is the Erdős-Rényi result. For $k=2$ and $R = \mathbb{Z}_2$ the theorem was proved in [3]. Our approach to the general case combines the method of [3] with some additional new ideas.

The case $p = \frac{k \log n - \omega(n)}{n}$ of Theorem 1.1 is straightforward: Let $g(Y)$ denote the number of isolated $(k-1)$-simplices of Y. Then

$$E[g] = \binom{n}{k}(1-p)^{n-k} = \Omega(\exp(\omega(n))) \ \ .$$

A standard second moment argument then shows that

$$\Pr[\ H^{k-1}(Y) = 0 \ ] \leq \Pr[\ g = 0 \ ] = o(1) \ .$$

The case $p = \frac{k \log n + \omega(n)}{n}$ is more involved. For a $\phi \in C^{k-1}(\Delta_{n-1})$ denote by $[\phi]$ the image of ϕ in $H^{k-1}(\Delta_{n-1}^{(k-1)})$. Let

$$b(\phi) = |\{\tau \in \Delta_{n-1}(k) \ : \ d_{k-1}\phi(\tau) \neq 0\}| \ .$$

For any complex $Y \supset \Delta_{n-1}^{(k-1)}$ we identify $H^{k-1}(Y)$ with its image under the natural injection $H^{k-1}(Y) \to H^{k-1}(\Delta_{n-1}^{(k-1)})$. It follows that for $\phi \in C^{k-1}(\Delta_{n-1})$

$$\Pr[\ [\phi] \in H^{k-1}(Y) \ ] = (1-p)^{b(\phi)} \ .$$

For $\phi \in C^{k-1}(\Delta_{n-1})$ let $\mathrm{supp}(\phi) = \{\sigma \in \Delta_{n-1}(k-1) : \phi(\sigma) \neq 0\}$. The *weight* of such ϕ is defined by

$$w(\phi) = \min \ \{ \ |\mathrm{supp}(\phi')| \ : \ \phi' \in C^{k-1}(\Delta_{n-1}) \ , \ [\phi'] = [\phi] \ \} =$$

$$\min \ \{ \ |\mathrm{Supp}(\phi + d_{k-2}\psi)| \ : \ \psi \in C^{k-2}(\Delta_{n-1}) \ \}.$$

A k-uniform hypergraph $\mathcal{F} \subset \binom{[n]}{k}$ is *connected* if for any $\sigma, \tau \in \mathcal{F}$ there exists a sequence $\sigma = \sigma_0, \ldots, \sigma_t = \tau \in \mathcal{F}$ such that $|\sigma_i \cap \sigma_{i-1}| = k-1$ for all $1 \leq i \leq t$. Let

$$\mathcal{G}_n = \{0 \neq \phi \in C^{k-1}(\Delta_{n-1}) \ : \ \mathrm{supp}(\phi) \ \text{ is connected} \ , \ w(\phi) = |\mathrm{supp}(\phi)|\} \ .$$

If $H^{k-1}(Y) \neq 0$ and $\phi \in C^{k-1}(\Delta_{n-1})$ is a cochain of minimum support size such that $0 \neq [\phi] \in H^{k-1}(Y)$, then $\phi \in \mathcal{G}_n$. Therefore

$$\Pr[\ H^{k-1}(Y) \neq 0 \ ] \leq \sum_{\phi \in \mathcal{G}_n} \Pr[\ [\phi] \in H^{k-1}(Y) \ ] = \sum_{\phi \in \mathcal{G}_n} (1-p)^{b(\phi)} \ .$$

Theorem 1.1 will thus follow from

Theorem 1.2. *For* $p = \frac{k \log n + \omega(n)}{n}$

$$\sum_{\phi \in \mathcal{G}_n} (1-p)^{b(\phi)} = o(1) \quad . \tag{2}$$

The main ingredients in the proof of Theorem 1.2 are a lower bound on $b(\phi)$ given in Section 2, and an estimate for the number of $\phi \in \mathcal{G}_n$ with prescribed values of $b(\phi)$ given in Section 3. In Section 4 we combine these results to derive Theorem 1.2. The group R and the dimension k are fixed throughout the paper. We use $c_i = c_i(r,k)$ to denote constants depending on r and k alone.

2 A lower bound on $b(\phi)$

We bound $b(\phi)$ in terms of the weight $w(\phi)$.

Proposition 2.1. *For* $\phi \in C^{k-1}(\Delta_{n-1})$

$$b(\phi) \geq \frac{n w(\phi)}{k+1} \quad . \tag{3}$$

Proof: For an ordered simplex $\tau = [v_0, \ldots, v_\ell]$ and a vertex $v \notin \tau$, let $v\tau = [v, v_0, \ldots, v_\ell]$. For $u \in V$ define $\phi_u \in C^{k-2}(\Delta_{n-1})$ by

$$\phi_u(\tau) = \begin{cases} \phi(u\tau) & u \notin \tau \\ 0 & u \in \tau \end{cases} . \tag{4}$$

Let $\sigma \in \Delta_{n-1}(k-1)$ and $u \in V$. Then

$$\phi(\sigma) - d_{k-2}\phi_u(\sigma) = \begin{cases} d_{k-1}\phi(u\sigma) & u \notin \sigma \\ 0 & u \in \sigma \end{cases} .$$

It follows that

$$(k+1)|\mathrm{supp}(d_{k-1}\phi)| = |\{(\tau, u) : u \in \tau \in \mathrm{supp}(d_{k-1}\phi)\}| =$$

$$|\{(\sigma, u) \in \Delta_{n-1}(k-1) \times V \ : \ \sigma \in \mathrm{supp}(\phi - d_{k-2}\phi_u)\}| =$$

$$\sum_{u \in V} |\mathrm{supp}(\phi - d_{k-2}\phi_u)| \geq n w(\phi) \quad .$$

□

Remark: The following example shows that equality can be attained in (3). Let n be divisible by $k+1$, and let $[n] = \cup_{i=0}^{k} V_i$ be a partition of $[n]$ with $|V_i| = \frac{n}{k+1}$. Consider the unique cochain $\phi \in C^{k-1}(\Delta_{n-1})$ that satisfies

$$\phi([v_0, \ldots, v_{k-1}]) = \begin{cases} 1 & v_i \in V_i \text{ for all } 0 \leq i \leq k-1 \\ 0 & |\{v_0, \ldots, v_{k-1}\} \cap V_i| \neq 1 \text{ for some } 0 \leq i \leq k-1. \end{cases}$$

Then $b(\phi) = (\frac{n}{k+1})^{k+1}$, and it can be shown that $w(\phi) = |\mathrm{supp}(\phi)| = (\frac{n}{k+1})^k$.

3 The number of ϕ with prescribed $b(\phi)$

Let

$$\mathcal{G}_n(m) = \{\phi \in \mathcal{G}_n : |\mathrm{supp}(\phi)| = m\}$$

and for $0 \leq \theta \leq 1$ let

$$\mathcal{G}_n(m, \theta) = \{\phi \in \mathcal{G}_n(m) : b(\phi) = (1-\theta)mn\} .$$

Write $g_n(m) = |\mathcal{G}_n(m)|$ and $g_n(m, \theta) = |\mathcal{G}_n(m, \theta)|$. Proposition 2.1 implies that $g_n(m, \theta) = 0$ for $\theta > \frac{k}{k+1}$. Our main estimate is the following

Proposition 3.1. *There exists a constant* $c_1 = c_1(r, k)$ *such that for any* $n \geq 10k^2$, $m \geq \frac{n}{2k}$, *and* $\theta \geq \frac{1}{2k}$

$$g_n(m, \theta) \leq \left(c_1 \cdot n^{(k-1)(1-\theta(1-\frac{1}{2k^2}))}\right)^m \quad . \tag{5}$$

The proof of Proposition 3.1 depends on a certain partial domination property of hypergraphs. Let $\mathcal{F} \subset \binom{[n]}{k}$ be a k-uniform hypergraph of cardinality $|\mathcal{F}| = m$. For $\sigma \in \mathcal{F}$ let

$$\beta_{\mathcal{F}}(\sigma) = |\{\tau \in \binom{[n]}{k+1} : \binom{\tau}{k} \cap \mathcal{F} = \{\sigma\}\}|$$

and let $\beta(\mathcal{F}) = \sum_{\sigma \in \mathcal{F}} \beta_{\mathcal{F}}(\sigma)$. Clearly $\beta_{\mathcal{F}}(\sigma) \leq n - k$ and $\beta(\mathcal{F}) \leq m(n-k)$. For $S \subset \mathcal{F}$ let

$$\Gamma(S) = \{\eta \in \mathcal{F} : |\eta \cap \sigma| = k - 1 \text{ for some } \sigma \in S\} \quad .$$

Claim 3.2. *Let* $0 < \epsilon \leq \frac{1}{2}$ *and* $n > 2 \log \frac{1}{\epsilon} + k$. *Suppose that*

$$\beta(\mathcal{F}) \leq (1 - \theta) m(n - k)$$

for some $0 < \theta \leq 1$. *Then there exists a subfamily* $S \subset \mathcal{F}$ *such that*

$$|\Gamma(S)| \geq (1 - \epsilon)\theta m$$

and

$$|S| < (20 \log \frac{1}{\epsilon}) \cdot \frac{m}{n-k} + 2 \log \frac{1}{\epsilon\theta} \quad .$$

proof: Let $c(\epsilon) = 2 \log \frac{1}{\epsilon}$. Choose a random subfamily $S \subset \mathcal{F}$ by picking each $\sigma \in \mathcal{F}$ independently with probability $\frac{c(\epsilon)}{n-k}$. For any $\sigma \in \mathcal{F}$ there exist distinct $v_1, \ldots, v_{n-k-\beta_{\mathcal{F}}(\sigma)} \in [n] - \sigma$ and $\tau_1, \ldots, \tau_{n-k-\beta_{\mathcal{F}}(\sigma)} \in \binom{\sigma}{k-1}$ such that $\tau_i \cup \{v_i\} \in \mathcal{F}$ for all i. In particular

$$\Pr[\ \sigma \notin \Gamma(S)\ ] \leq \left(1 - \frac{c(\epsilon)}{n-k}\right)^{n-k-\beta_{\mathcal{F}}(\sigma)} \quad ,$$

hence

$$E[\ |\mathcal{F} - \Gamma(S)|\ ] \leq \sum_{\sigma \in \mathcal{F}} \left(1 - \frac{c(\epsilon)}{n-k}\right)^{n-k-\beta_{\mathcal{F}}(\sigma)} \quad . \tag{6}$$

Since

$$\sum_{\sigma\in\mathcal{F}}(n-k-\beta_{\mathcal{F}}(\sigma)) = m(n-k)-\beta(\mathcal{F}) \geq \theta m(n-k)$$

it follows by convexity from (6) that

$$E[\ |\mathcal{F}-\Gamma(S)|\ ] \leq (1-\theta)m + \theta m\Big(1-\frac{c(\epsilon)}{n-k}\Big)^{n-k} \leq$$

$$(1-\theta)m + \theta m e^{-c(\epsilon)} = (1-\theta)m + \theta m\epsilon^2 \quad .$$

Therefore

$$E[\ |\Gamma(S)|\ ] \geq (1-\epsilon^2)\theta m \quad .$$

Hence, since $|\Gamma(S)| \leq |\mathcal{F}| = m$, it follows that

$$\Pr[\ |\Gamma(S)| \geq (1-\epsilon)\theta m\ ] > \epsilon(1-\epsilon)\theta \quad . \tag{7}$$

On the other hand

$$E[\ |S|\ ] = \frac{c(\epsilon)m}{n-k}$$

and by the large deviation inequality (see e.g. Theorem A.1.12 in [1])

$$\Pr[\ |S| > \lambda\frac{c(\epsilon)m}{n-k}\ ] < \Big(\frac{e}{\lambda}\Big)^{\lambda\frac{c(\epsilon)m}{n-k}} \tag{8}$$

for all $\lambda \geq 1$. Let

$$\lambda = 10 + \frac{n-k}{m}\Big(\frac{\log\frac{1}{\theta}}{\log\frac{1}{\epsilon}}+1\Big)$$

then

$$\epsilon(1-\epsilon)\theta > \Big(\frac{e}{\lambda}\Big)^{\lambda\frac{c(\epsilon)m}{n-k}} \quad .$$

Hence by (7) and (8) there exists an $S \subset \mathcal{F}$ such that $|\Gamma(S)| \geq (1-\epsilon)\theta m$ and

$$|S| \leq \lambda\frac{c(\epsilon)m}{n-k} = (20\log\frac{1}{\epsilon})\cdot\frac{m}{n-k} + 2\log\frac{1}{\epsilon\theta} \quad .$$

□

Proof of Proposition 3.1: Define

$$\mathcal{F}_n(m,\theta) = \{\mathcal{F} \subset \binom{[n]}{k} \ : \ |\mathcal{F}| = m \ , \ \beta(\mathcal{F}) \leq (1-\theta)mn\}$$

and let $f_n(m,\theta) = \big|\mathcal{F}_n(m,\theta)\big|$. If $\phi \in \mathcal{G}_n(m,\theta)$, then $\mathcal{F} = \mathrm{Supp}(\phi) \in \mathcal{F}_n(m,\theta)$. Indeed, if $\tau \in \binom{[n]}{k+1}$ satisfies $\binom{\tau}{k} \cap \mathcal{F} = \{\sigma\}$, then $d_{k-1}\phi(\tau) = \pm\phi(\sigma) \neq 0$, hence $\beta(\mathcal{F}) \leq b(\phi) = (1-\theta)mn$. Therefore

$$g_n(m,\theta) \leq (r-1)^m f_n(m,\theta) \quad .$$

We next estimate $f_n(m,\theta)$. Let $\mathcal{F} \in \mathcal{F}_n(m,\theta)$, then

$$\beta(\mathcal{F}) \leq (1-\theta)mn = (1-\frac{\theta n - k}{n-k})m(n-k) \ .$$

Applying Claim 3.2 with $\theta' = \frac{\theta n - k}{n-k}$ and $\epsilon = \frac{1}{2k^2}$, it follows that there exists an $S \subset \mathcal{F}$ of cardinality $|S| \leq \frac{c_2 m}{n}$ with $c_2 = c_2(k)$, such that $|\Gamma(S)| \geq (1-\frac{1}{2k^2})\theta' m$. The injectivity of the mapping

$$\mathcal{F} \rightarrow (S, \Gamma(S), \mathcal{F} - \Gamma(S))$$

implies that

$$f_n(m,\theta) \leq \sum_{i=0}^{c_2 m/n} \binom{\binom{n}{k}}{i} \cdot 2^{(\frac{c_2 m}{n})kn} \cdot \sum_{j=0}^{(1-\theta'(1-\frac{1}{2k^2}))m} \binom{\binom{n}{k}}{j} \leq$$

$$c_3^m \binom{\binom{n}{k}}{(1-\theta'(1-\frac{1}{2k^2}))m} \leq$$

$$c_4^m \left(\frac{n^k}{m}\right)^{(1-\theta'(1-\frac{1}{2k^2}))m} \quad .$$

Therefore

$$g_n(m,\theta) \leq (r-1)^m f_n(m,\theta) \leq$$

$$(r-1)^m c_4^m \left(\frac{n^k}{m}\right)^{(1-\theta'(1-\frac{1}{2k^2}))m} \leq$$

$$\left(c_1 \cdot n^{(k-1)(1-\theta(1-\frac{1}{2k^2}))}\right)^m \quad .$$

□

4 Proof of Theorem 1.2

Proof of Theorem 1.2: Let $\omega(n) \to \infty$ and let $p = \frac{k \log n + \omega(n)}{n}$. We have to show that

$$\sum_{m \geq 1} \sum_{\phi \in \mathcal{G}_n(m)} (1-p)^{b(\phi)} = o(1) \quad . \tag{9}$$

We deal separately with two intervals of m :

(i) $1 \leq m \leq \frac{n}{2k}$. If $\phi \in \mathcal{G}_n(m)$ then $\mathrm{supp}(\phi) \subset \binom{[n]}{k}$ is a connected k-uniform hypergraph, hence there exists a subset $S \subset [n]$ of cardinality $|S| \leq m+k-1$ such that $\mathrm{supp}(\phi) \subset \binom{S}{k}$. Since $d_{k-1}\phi(u\sigma) = \phi(\sigma) \neq 0$ for any $\sigma \in \mathrm{supp}(\phi)$ and $u \notin S$, it follows that $b(\phi) \geq m(n-m-k+1)$. The trivial estimate

$$g_n(m) \leq (r-1)^m \binom{\binom{n}{k}}{m} \leq c_5^m \left(\frac{n^k}{m}\right)^m$$

implies that for $n \geq 6k$

$$g_n(m)(1-p)^{m(n-m-k+1)} \leq$$

$$c_5^m \frac{n^{km}}{m^m}\left(1 - \frac{k \log n + w(n)}{n}\right)^{m(n-m-k+1)} \leq$$

$$c_5^m \frac{n^{km}}{m^m} n^{\frac{-k(n-m-k+1)m}{n}} e^{\frac{-w(n)(n-m-k+1)m}{n}} \leq$$

$$c_6^m \left(\frac{n^k}{m} n^{\frac{-k(n-m)}{n}}\right)^m e^{-\frac{w(n)}{3}m} =$$

$$(c_6 \frac{n^{\frac{km}{n}}}{m} e^{-\frac{w(n)}{3}})^m \quad .$$

Since

$$\frac{n^{\frac{km}{n}}}{m} \leq \begin{cases} n^{kn^{-1/3}} & m \leq n^{2/3} \\ n^{-1/6} & n^{2/3} \leq m \leq \frac{n}{2k} \end{cases}$$

it follows that there exists a $c_7 = c_7(r,k)$ such that for $m \leq \frac{n}{2k}$ and $n \geq 6k$

$$g_n(m)(1-p)^{m(n-m-k+1)} \leq \left(c_7 e^{-\frac{w(n)}{3}}\right)^m \quad .$$

Therefore

$$\sum_{m=1}^{n/2k} \sum_{\phi \in \mathcal{G}_n(m)} (1-p)^{b(\phi)} \leq \sum_{m=1}^{n/2k} g_n(m)(1-p)^{m(n-m-k+1)} \leq$$

$$\sum_{m=1}^{n/2k} \Big(c_7 e^{-\frac{w(n)}{3}} \Big)^m = O(e^{-\frac{w(n)}{3}}) = o(1) \quad . \tag{10}$$

(ii) $m \geq \frac{n}{2k}$. Then

$$\sum_{m \geq n/2k} \sum_{\theta \leq 1/2k} \sum_{\phi \in \mathcal{G}_n(m,\theta)} (1-p)^{b(\phi)} =$$

$$\sum_{m \geq n/2k} \sum_{\theta \leq 1/2k} g_n(m,\theta)(1-p)^{(1-\theta)mn} \leq$$

$$\sum_{m \geq n/2k} g_n(m)(1-p)^{(1-\frac{1}{2k})mn} \leq$$

$$\sum_{m \geq n/2k} \Big(\frac{c_5 n^k}{m} \Big)^m n^{-(1-\frac{1}{2k})km} \leq$$

$$\sum_{m \geq n/2k} (2kc_5 n^{k-1})^m n^{-(1-\frac{1}{2k})km} =$$

$$\sum_{m \geq n/2k} \Big(2kc_5 n^{-1/2} \Big)^m = n^{-\Omega(n)} \quad . \tag{11}$$

Next note that by Proposition 2.1, $g_n(m,\theta) = 0$ for $\theta > \frac{k}{k+1}$. Hence, by Proposition 3.1

$$\sum_{m \geq n/2k} \sum_{\theta \geq 1/2k} \sum_{\phi \in \mathcal{G}_n(m,\theta)} (1-p)^{b(\phi)} =$$

$$\sum_{m \geq n/2k} \sum_{\theta \geq 1/2k} g_n(m,\theta)(1-p)^{(1-\theta)mn} \leq$$

$$\sum_{m \geq n/2k} \sum_{\substack{\theta \geq 1/2k \\ g_n(m,\theta) \neq 0}} \Big(c_1 \cdot n^{(k-1)(1-\theta(1-\frac{1}{2k^2}))} \Big)^m \cdot n^{-(1-\theta)km} =$$

$$\sum_{m \geq n/2k} \sum_{\substack{\theta \geq 1/2k \\ g_n(m,\theta) \neq 0}} \left(c_1 \cdot n^{\theta(1+\frac{k-1}{2k^2})-1}\right)^m \leq n^{k+1} \sum_{m \geq n/2k} \left(c_1 \cdot n^{\frac{k}{k+1}(1+\frac{k-1}{2k^2})-1}\right)^m =$$

$$n^{k+1} \sum_{m \geq n/2k} \left(c_1 n^{-1/2k}\right)^m = n^{-\Omega(n)} \quad . \tag{12}$$

Finally (9) follows from (10), (11) and (12).

□

5 Concluding Remarks

We have shown that in the model $Y_k(n,p)$ of random k-complexes on n vertices, the threshold for the vanishing of $H^{k-1}(Y;R)$ occurs at $p = \frac{k \log n}{n}$, provided that both k and the finite coefficient group R are fixed. One natural concrete question is whether $p = \frac{k \log n}{n}$ is also the threshold for the vanishing of $H^{k-1}(Y;\mathbb{Z})$.

More generally, in view of the detailed understanding of the evolution of random graphs (see e.g. [1]), it would be interesting to formulate and prove analogous statements concerning the topology of random complexes. For example, what is the higher dimensional counterpart of the remarkable double-jump phenomenon that occurs in random graphs?

ACKNOWLEDGMENT
We would like to thank Nati Linial for helpful discussions and comments.

References

[1] N. Alon and J. Spencer, *The Probabilistic Method*, 2nd Edition, Wiley-Intescience, 2000.

[2] P. Erdős and A. Rényi, On the evolution of random graphs, *Publ. Math. Inst. Hungar. Acad. Sci.* **5**(1960) 17-61.

[3] N. Linial and R. Meshulam, Homological connectivity of random 2-complexes, *Combinatorica*, to appear.

[4] J. Munkres, *Elements of Algebraic Topology*, Addison-Wesley, 1984.