Random Latin Squares and 2-Dimensional Expanders

Alexander Lubotzky[*]   Roy Meshulam[†]

December 17, 2014

Abstract

Expander graphs have been playing an important role in combinatorics and computer science over the last four decades. In recent years a theory of high dimensional expanders is emerging, but as of now all known examples of expanders (random and explicit) have unbounded degrees. The question of existence of bounded degree high dimensional expanders was raised by Gromov and by Dotterrer and Kahle. In this paper we present a new model, based on Latin squares, of 2-dimensional complexes of bounded edge degrees that are expanders with probability tending to 1.

1 Introduction

The notion of expander graphs has been extremely useful in computer science, combinatorics and even pure mathematics (see [10, 15] and the references therein). In recent years there is a growing interest in high-dimensional expanders (see the survey [16]). The k-dimensional version of the graphical Cheeger constant, called "coboundary expansion", came up independently in the work of Linial, Meshulam and Wallach [12, 19] on homological connectivity of random complexes and in Gromov's remarkable work [7, 3] where it is

shown that this notion of expansion implies the topological overlap property (see Section 6).

We recall some topological terminology. Let X be a simplicial complex on the vertex set V. For $k \geq 0$, let $X^{(k)}$ denote the k-dimensional skeleton of X and let $X(k)$ be the family of k-dimensional faces of X. Let $D_k(X)$ be the maximum number of $(k+1)$-dimensional faces of X containing a common k-dimensional face. Let $C^k(X; \mathbb{F}_2)$ denote the space of $\mathbb{F}_2$-valued k-cochains of X. The k-coboundary map $d_k : C^k(X; \mathbb{F}_2) \to C^{k+1}(X; \mathbb{F}_2)$ is given

$$d_k \phi(v_0, \cdots, v_{k+1}) = \sum_{i=0}^{k+1} \phi(v_0, \ldots, v_{i-1}, v_{i+1}, \ldots, v_{k+1}).$$

It will be convenient to augment the cochain complex $\{C^i(X; \mathbb{F}_2)\}_{i=0}^{\infty}$ with a (-1)-degree term $C^{-1}(X; \mathbb{F}_2) = \mathbb{F}_2$ with a coboundary map $d_{-1} : C^{-1}(X; \mathbb{F}_2) \to C^0(X; \mathbb{F}_2)$ given by $d_{-1}(a)(v) = a$ for $a \in \mathbb{F}_2, v \in V$.

Let $Z^k(X; \mathbb{F}_2) = \ker d_k$ be the space of $\mathbb{F}_2$-valued k-cocycles of X and let $B^k(X; \mathbb{F}_2) = d_{k-1}(C^{k-1}(X; \mathbb{F}_2))$ be the space of $\mathbb{F}_2$-valued k-coboundaries of X. The k-th reduced cohomology group of X with $\mathbb{F}_2$ coefficients is

$$\tilde{H}^k(X; \mathbb{F}_2) = \frac{Z^k(X; \mathbb{F}_2)}{B^k(X; \mathbb{F}_2)}.$$

For $\phi \in C^k(X; \mathbb{F}_2)$, let $[\phi]$ denote the image of ϕ in the quotient space $C^k(X; \mathbb{F}_2)/B^k(X; \mathbb{F}_2)$. Let

$$\|\phi\| = |\{\sigma \in X(k) : \phi(\sigma) \neq 0\}| = |\mathrm{supp}(\phi)|$$

and

$$\|[\phi]\| = \min\{|\mathrm{supp}(\phi + d_{k-1}\psi)| : \psi \in C^{k-1}(X; \mathbb{F}_2)\}.$$

We will sometimes write $\|\phi\|_X$ in case of ambiguity concerning X.

Definition. *The k-th coboundary expansion constant of X (see [12, 19, 7, 2]) is defined by*

$$h_k(X) = \min\left\{\frac{\|d_k \phi\|}{\|[\phi]\|} : \phi \in C^k(X; \mathbb{F}_2) - B^k(X; \mathbb{F}_2)\right\}.$$

A complex X is a (k, d, ϵ)-expander if

$$D_{k-1}(X) \leq d \quad and \quad h_{k-1}(X) \geq \epsilon.$$

2

Remarks:
1. Note that $h_k(X) = 0$ iff $\tilde{H}^k(X; \mathbb{F}_2) \neq 0$.
2. Let $G = (V, E)$ be a graph. For $S \subset V$ let

$$E(S, V - S) = \{e \in E : |e \cap S| = 1\}$$

be the cut determined by S and let $e(S, V - S) = |E(S, V - S)|$. Viewing G as a 1-dimensional complex, it is easy to check (see [2]) that $h_0(G)$ coincides with the Cheeger constant of G:

$$h_0(G) = \min_{0 < |S| \le \frac{|V|}{2}} \frac{e(S, V - S)}{|S|}.$$

Questions concerning existence and construction of families of expander graphs have been the subject of intense research for the last 40 years. Clearly the complete graph is an expander. Using random methods it is also not difficult to show the existence of infinite families of $(1, d, \epsilon)$-expander graphs for some fixed d and $\epsilon > 0$. Explicit constructions are considerably deeper and involve applications of Kazhdan Property (T), the Ramanujan conjecture and the Zigzag product (see [14, 10]).

In this paper we are concerned with the existence of higher dimensional expanders. Here even the fact that the full k-skeleton of the $(n-1)$-simplex Δ_{n-1} is expanding is not completely obvious. In [19, 7] it was shown that the expansion of Δ_{n-1} is given by

$$h_{k-1}(\Delta_{n-1}) = \frac{n}{k+1}. \tag{1}$$

We next consider the expansion of random complexes in the following natural k-dimensional extension of the Erdős-Rényi random graph model (see [12, 19]). For $k \ge 1$ and $0 \le p \le 1$, let $Y_k(n, p)$ denote the probability space of all k-dimensional random subcomplexes of Δ_{n-1} obtained by starting with the full $(k-1)$-dimensional skeleton of Δ_{n-1} and then adding each k-simplex independently with probability p. Using the Chernoff bound it directly follows from (1) that there exists a constant $C(k) = \Theta(k^2)$ such that if $p = \frac{C(k) \log n}{n}$ then $Y \in Y_k(n, p)$ is asymptotically almost surely (a.a.s.) a $(k, \Theta(\log n), 1)$-expander (see [8, 2]). This of course implies that $Y \in Y_k(n, p)$ a.a.s. satisfies $H_{k-1}(Y; \mathbb{F}_2) = 0$. Obtaining the exact probability threshold $p = \frac{k \log n}{n}$ for the vanishing of $H_{k-1}(Y; \mathbb{F}_2)$ is more involved (see [12, 19]).

3

Dotterrer and Kahle [2] proved results analogous to (1) for skeleta of cross polytopes and for complete multipartite complexes with similar consequences for the expansion of their random subcomplexes. As in the case of Δ_{n-1}, all these complexes have unbounded degrees.

The question of existence of bounded degree high dimensional expanders is more complicated. For example, the random model of bounded (vertex) degree 2-dimensional complexes used in [3] for geometric overlapping does not give rise to expanders. In fact, these complexes have non-trivial rational 1-cohomology with probability tending to 1. Similarly, the Ramanujan complexes [17] can have non-trivial 1-cohomology with $\mathbb{F}_2$ coefficients.

Here we establish for the first time the existence of an infinite family of 2-dimensional expanders of bounded (edge) degree d, i.e. $(2, d, \epsilon)$-expanders for some fixed d and $\epsilon > 0$, answering a question asked implicitly in [7] and explicitly in [2]. Our proof is probabilistic and depends on the following new model, based on Latin squares, of random 2-dimensional simplicial complexes with bounded edge degrees.

We introduce some notation and definitions that will be used throughout the paper. Let $\mathbb{S}_n$ be the symmetric group on $[n] = \{1, \ldots, n\}$. A k-tuple $(\pi_1, \ldots, \pi_k) \in \mathbb{S}_n^k$ is *legal* if $\pi_i \pi_j^{-1}$ is fixed point free for all $1 \leq i < j \leq k$. A *Latin Square* of order n is a legal n-tuple of permutations $L = (\pi_1, \ldots, \pi_n) \in \mathbb{S}_n^n$. Let $\mathcal{L}_n$ denote the uniform probability space of all Latin squares of order n. Let $V_1 = \{a_i\}_{i=1}^n, V_2 = \{b_i\}_{i=1}^n, V_3 = \{c_i\}_{i=1}^n$ be three disjoint sets. The complete 3-partite complex $T_n = V_1 * V_2 * V_3$ consists of all $\sigma \subset V = V_1 \cup V_2 \cup V_3$ such that $|\sigma \cap V_i| \leq 1$ for $1 \leq i \leq 3$. An $L = (\pi_1, \ldots, \pi_n) \in \mathcal{L}_n$ determines a subcomplex $T_n^{(1)} \subset Y(L) \subset T_n$ whose 2-simplices are $[a_i, b_j, c_{\pi_i(j)}]$ where $1 \leq i, j \leq n$. In particular $Y(L)$ has $3n^2$ edges and every edge lies in a unique 2-simplex, i.e. $D_1(Y(L)) = 1$. Fix d and regard $\mathcal{L}_n^d$ as a uniform probability space. For $\underline{L}^d = (L_1, \ldots, L_d) \in \mathcal{L}_n^d$, let $Y(\underline{L}^d) = \cup_{i=1}^d Y(L_i)$. Note that $D_1(Y(\underline{L}^d)) \leq d$. Let $\mathcal{Y}(n, d)$ denote the probability space of all complexes $Y(\underline{L}^d)$ with measure induced from $\mathcal{L}_n^d$.

Theorem 1. *There exist $\epsilon > 0, d < \infty$ such that*

$$\lim_{n \to \infty} \Pr\left[Y \in \mathcal{Y}(n, d) : h_1(Y) > \epsilon\right] = 1.$$

Theorem 1 is proved in two steps: in the first it is shown that 1-cochains with small support have large coboundaries. This part is inspired by (and is in fact easier than) the results of [11]. To handle the case of 1-cochains

with large support we have to establish a concentration of measure theorem for the space $\mathcal{L}_n$. This result may be viewed as a Latin square (partial) analogue of Maurey's large deviations bound for Lipschitz functions on the symmetric group [18] (see also Theorem 7.5 in [20] and Theorem 7.4.2 in [1]). Maurey's proof uses martingales and depends on a natural filtration of $\mathbb{S}_n$ by subgroups. In the absence of such structure for Latin squares, we had to develop a different approach based on permanent inequalities (for further comments see Section 4).

The paper is organized as follows. In Section 2 we describe the strategy of the proof. In Section 3 we prove a lower bound (Proposition 2) on the expansion of small cochains in terms of the spectral gaps of the vertex links. In Section 4 we prove a large deviations bound for random Latin squares (Theorem 4). This result is the key for showing expansion of large cochains. In Section 5 we use the above mentioned results to obtain Theorem 1. We conclude in Section 6 with some questions and comments.

2   Outline of the proof

In order to prove Theorem 1 we have to show the existence of fixed d and $\epsilon > 0$ such that for almost all $Y \in \mathcal{Y}(n, d)$, all $\phi \in C^1(T_n; \mathbb{F}_2)$ satisfy $\|d_1\phi\|_Y \geq \epsilon\|[\phi]\|$.

Let $0 < c < 1$ be a constant whose value will be assigned later. A 1-cochain ϕ is called c-*small* if $\|[\phi]\| \leq cn^2$ and c-*large* otherwise. We first address the expansion of c-small 1-cochains. Let Y be a subcomplex of T_n with maximum edge degree at most d. The link $Y_v = \mathrm{lk}(Y, v)$ of a vertex $v \in V$ is a bipartite graph. Let $\mu(Y_v)$ be the second smallest eigenvalue of the Laplacian of Y_v and let $\tilde{\mu} = \min_{v \in V} \mu(Y_v)$. Inspired by the results of [11], we show in Proposition 2 that

$$\|d_1\phi\|_Y \geq \left(\frac{(1 - c^{1/3})\tilde{\mu}}{2} - \frac{d}{3}\right)\|[\phi]\|. \tag{2}$$

Suppose now that $Y = Y(\underline{L}^d) \in \mathcal{Y}(n, d)$. As each Y_v is a random bipartite graph of maximum degree d, it follows from a theorem of Friedman [4, 5] that $\mu(Y_v) \geq d - O(\sqrt{d})$ for all $v \in V$ with probability $1 - o(1)$. Hence by (2) there exist $c > 0$ and d such that for almost all $Y \in \mathcal{Y}(n, d)$

$$\|d_1\phi\|_Y \geq \|[\phi]\| \tag{3}$$

for all c-small ϕ's.

We next consider the expansion of c-large 1-cochains. Here, in contrast with the first case, we bound the probability of non-expansion separately for each cochain. Let $\phi \in C^1(T_n; \mathbb{F}_2)$ be c-large. A result of Gromov (Section 2.13 in [7]) asserts that $h_1(T_n) = \Omega(n)$. Gromov's bound was improved by Dotterrer and Kahle (Proposition 5.7 in [2]) who showed that $h_1(T_n) \geq \frac{n}{5}$. It follows that

$$\mathcal{E} = \{\sigma \in T_n(2) : d_1\phi(\sigma) \neq 0\}$$

satisfies

$$|\mathcal{E}| = \|d_1\phi\|_{T_n} \geq \frac{n}{5}\|[\phi]\| \geq \frac{cn^3}{5}.$$

For a Latin square L in the uniform probability space $\mathcal{L}_n$ let $f_{\mathcal{E}}(L) = |Y(L)(2) \cap \mathcal{E}|$ be the number of 2-simplices in $Y(L)$ that belong to $\mathcal{E}$. As $|Y(L)(2)| = n^2$ and $|T_n(2)| = n^3$, it follows by homogeneity that the expectation of $f_{\mathcal{E}}$ over $\mathcal{L}_n$ is

$$E[f_{\mathcal{E}}] = \frac{|\mathcal{E}|}{n} \geq \frac{cn^2}{5}.$$

In Theorem 4 we prove a large deviation type bound for $f_{\mathcal{E}}$ showing that

$$\Pr[f_{\mathcal{E}} < 10^{-5}c^2n^2] < e^{-\delta n^2}$$

for some $\delta = \delta(c) > 0$. Now let $\underline{L}^d = (L_1, \ldots, L_d) \in \mathcal{L}_n^d$. Since

$$\|d_1\phi\|_{Y(\underline{L}^d)} = |Y(\underline{L}^d)(2) \cap \mathcal{E}|$$
$$\geq \max_{1 \leq i \leq d} |Y(L_i)(2) \cap \mathcal{E}| = \max_{1 \leq i \leq d} f_{\mathcal{E}}(L_i)$$

it follows that

$$\Pr\left[\underline{L}^d \in \mathcal{L}_n^d : \|d_1\phi\|_{Y(\underline{L}^d)} < 10^{-5}c^2n^2\right] < e^{-\delta dn^2}. \tag{4}$$

As the number of 1-cochains is 2^{3n^2}, Eq. (4) implies that the probability that there exists a large 1-cochain ϕ with $\|d_1\phi\|_{Y(\underline{L}^d)} < 10^{-5}c^2n^2$ is at most $2^{3n^2}e^{-\delta dn^2}$. Choosing a sufficiently large d and noting that $\|[\phi]\| \leq 3n^2$ for all ϕ, it follows that

$$\Pr\left[\underline{L}^d \in \mathcal{L}_n^d : \frac{\|d_1\phi\|_{Y(\underline{L}^d)}}{\|[\phi]\|} \geq 10^{-6}c^2 \ \text{ for all c-large } \phi\right] = 1 - o(1). \tag{5}$$

Theorem 1 is now a consequence of (3) and (5). In the next sections we prove Proposition 2 and Theorem 4 which are the key ingredients in the argument.

3 Expansion of Small Cochains

In this section we relate the expansion of small 1-cochains to the spectral gaps of the vertex links. The Laplacian of a graph $G = (V, E)$ is the $V \times V$ positive semidefinite matrix L_G given by

$$L_G(u, v) = \begin{cases} \deg(u) & u = v \\ -1 & uv \in E \\ 0 & \text{otherwise} \end{cases}$$

Let $0 = \mu_1(G) \leq \mu_2(G) \leq \cdots \leq \mu_{|V|}(G)$ denote the eigenvalues of G. The spectral gap of G is $\mu(G) = \mu_2(G)$.

Let Y be a subcomplex of the complete 3-partite complex T_n on the vertex set $V = V_1 \cup V_2 \cup V_3$ with a complete 1-skeleton $T_n^{(1)}$. The link $Y_v = \text{lk}(Y, v)$ of a vertex $v \in V_i$ is a bipartite graph on the vertex set $V_v = \text{lk}(Y, v)(0) = V_j \cup V_{j'}$ where $\{1, 2, 3\} = \{i, j, j'\}$. Let $d = \max_{e \in Y(1)} |\text{lk}(Y, e)|$ be the maximal degree of an edge in Y. Let $\tilde{\mu} = \min_{v \in V} \mu(Y_v)$.

Let $\phi \in C^1(Y; \mathbb{F}_2)$ be a 1-cochain of Y. We identify ϕ with the graph $G = (V, E)$ where $E = \{uv : \phi(uv) = 1\}$. For $v \in V$, let

$$S_v = \Gamma_G(v) = \{u \in V : uv \in E\} \subset V_v$$

and let $s_v = |S_v| = \deg_G(v)$.

Proposition 2. *Let $c < 1$ and suppose $m = \|[\phi]\| \leq cn^2$. Then*

$$\|d_1 \phi\| \geq \left(\frac{(1 - c^{1/3})\tilde{\mu}}{2} - \frac{d}{3} \right) \|[\phi]\|.$$

Proof: We may assume that $\|\phi\| = \|[\phi]\| = m$. Therefore $\|\phi\| \leq \|\phi + d_0 1_v\|$ and hence $s_v \leq \frac{|V_v|}{2} = n$ for all $v \in V$. We will need the following

Claim 3.
$$\sum_{v \in V} s_v^2 \leq (1 + 3c^{1/3})mn.$$

Proof: Let
$$I = \{v \in V : s_v \geq c^{1/3}n\}.$$

Then
$$|I| \leq \frac{\sum_{v \in V} s_v}{c^{1/3}n} = \frac{2m}{c^{1/3}n}.$$

Let
$$E_0 = \{uv \in E : s_u + s_v \geq (1 + c^{1/3})n\}.$$
If $uv \in E_0$ then $u, v \in I$. It follows that
$$|E_0| \leq \frac{|I|^2}{2} \leq \frac{2m^2}{c^{2/3}n^2} \leq 2c^{1/3}m.$$
Therefore
$$\begin{aligned}
\sum_{v \in V} s_v^2 = \sum_{uv \in E} (s_u + s_v) &= \sum_{uv \in E_0} (s_u + s_v) + \sum_{uv \in E - E_0} (s_u + s_v) \\
&\leq |E_0| \cdot 2n + (m - |E_0|) \cdot (1 + c^{1/3})n \\
&= (1 - c^{1/3})n|E_0| + (1 + c^{1/3})mn \\
&\leq (1 + 3c^{1/3})mn.
\end{aligned}$$

$\square$

Proof of Proposition 2: For $v \in V$, define $\phi_v \in C^0(Y; \mathbb{F}_2)$ by
$$\phi_v(u) = \begin{cases} \phi(vu) & uv \in Y(1) \\ 0 & otherwise \end{cases}$$
Note that if $uw \in \mathrm{lk}(Y, v)(1)$ then
$$d_1\phi(vuw) = \phi(uw) - d_0\phi_v(uw).$$
Additionally, if $v \in V$ then
$$\begin{aligned}
|\mathrm{supp}(d_0\phi_v) \cap \mathrm{lk}(Y, v)| &= |\{uw \in Y_v(1) : \phi(vu) \neq \phi(vw)\}| \\
&= |\{uw \in Y_v(1) : u \in S_v, w \in V_v - S_v\}| = e_{Y_v}(S_v, V_v - S_v).
\end{aligned}$$
Therefore
$$\begin{aligned}
3\|d_1\phi\| &= \sum_{v \in V} |\{(v, uw) \in V \times Y(1) : uw \in \mathrm{lk}(Y, v) ,\ d_1\phi(vuw) \neq 0\}| \\
&= \sum_{v \in V} |\mathrm{supp}(\phi - d_0\phi_v) \cap \mathrm{lk}(Y, v)| \\
&\geq \sum_{v \in V} |\mathrm{supp}(d_0\phi_v) \cap \mathrm{lk}(Y, v)| - \sum_{v \in V} |\mathrm{supp}(\phi) \cap \mathrm{lk}(Y, v)| \\
&= \sum_{v \in V} e_{Y_v}(S_v, V_v - S_v) - \sum_{uw \in \mathrm{supp}(\phi)} |\mathrm{lk}(Y, uw)|.
\end{aligned}$$

8

The Alon-Milman and Tanner bound (see Theorem 9.2.1 in [1]) asserts that

$$e_{Y_v}(S_v, V_v - S_v) \geq \frac{|S_v| \cdot |V_v - S_v|}{|V_v|} \mu(Y_v) = \frac{s_v(2n - s_v)}{2n} \mu(Y_v).$$

Combined with the assumption that $|\mathrm{lk}(Y, uw)| \leq d$ for all edges $uw \in T(1)$, it follows that

$$
\begin{aligned}
3\|d_1\phi\| &\geq \tilde{\mu} \sum_{v \in V} \frac{s_v(2n - s_v)}{2n} - md \\
&= \tilde{\mu}(2m - \frac{1}{2n} \sum_{v \in V} s_v^2) - md \\
&\geq \tilde{\mu}(2m - \frac{(1 + 3c^{1/3})mn}{2n}) - md \\
&= \left(\frac{3(1 - c^{1/3})\tilde{\mu}}{2} - d \right) m.
\end{aligned}
$$

$\square$

4   Large Deviations for Latin Squares

Let $0 < c \leq 1$ and let $\mathcal{E} \subset T_n(2)$ be a family of 2-simplices in T_n such that $|\mathcal{E}| \geq cn^3$. Define a random variable $f_{\mathcal{E}}$ on the space of Latin squares $\mathcal{L}_n$ by

$$f_{\mathcal{E}}(L) = |Y(L)(2) \cap \mathcal{E}|.$$

The expectation of $f_{\mathcal{E}}$ is

$$E[f_{\mathcal{E}}] = \frac{|\mathcal{E}|}{n} \geq cn^2.$$

In the next theorem we establish a one-sided tail estimate for the random variable $f_{\mathcal{E}}$. Let us remark that if instead of the space $\mathcal{L}_n$ we consider the larger probability space $\mathbb{S}_n^n$, then a similar estimate follows from Maurey's theorem [18, 20]. This however does not seem applicable to our case since the measure of $\mathcal{L}_n$ inside $\mathbb{S}_n^n$ is only $\exp(-\Omega(n^2))$.

Theorem 4. *There exists an $n_0(c)$ such that for all $n \geq n_0(c)$*

$$\Pr[f_{\mathcal{E}}(L) < 10^{-3}c^2n^2] < e^{-10^{-3}c^2n^2}.$$

In Subsection 4.1 we obtain an upper bound (Proposition 7) on the size of a certain family of permutations. This is the main ingredient in the proof of Theorem 4 given in Subsection 4.2.

4.1 Counting Restricted Permutations

Let $0 < \gamma \le 1/2$ and let E be a subset of $[n]^2$ such that $|E| \ge \gamma n^2$. Let $k \le \frac{\gamma n}{2}$ and let $F = (B_1, \ldots, B_n)$ be an n-tuple of subsets of $[n]$ such that $|B_i| = k$ for all $1 \le i \le n$. For a permutation $\pi \in \mathbb{S}_n$, let

$$g_E(\pi) = |\{(i, \pi(i))\}_{i=1}^n \cap E|. \tag{6}$$

Our first goal in this section is to bound the cardinality of the set

$$S(E, F) = \{\pi \in \mathbb{S}_n : g_E(\pi) \le \frac{\gamma n}{10} \ , \ \pi(i) \notin B_i \text{ for all } 1 \le i \le n\}.$$

It is instructive to first consider the case $k = 0$ where only the first restriction $g_E(\pi) \le \frac{\gamma n}{10}$ is relevant. The expectation of g_E over the uniform probability space $\mathbb{S}_n$ is $E[g_E] = \frac{|E|}{n} \ge \gamma n$, hence by Maurey's large deviation theorem [18] (see also Theorem 7.5 in [20] and Theorem 7.4.2 in [1])

$$\Pr\left[\pi \in \mathbb{S}_n : g_E(\pi) \le \frac{\gamma n}{10}\right]$$
$$\le \Pr\left[\pi \in \mathbb{S}_n : g_E(\pi) \le \frac{E[g_E]}{10}\right] < 2e^{-\frac{\gamma^2 n}{20}}.$$

Therefore if $k = 0$ then

$$|S(E, F)| \le 2n! \cdot e^{-\frac{\gamma^2 n}{20}} \le 2e\sqrt{n} \left(\frac{n}{e}\right)^n e^{-\frac{\gamma^2 n}{20}}. \tag{7}$$

Below we prove an extension of (7) for general $k \le \frac{\gamma n}{2}$ (under a mild assumption on E). Let us remark that the proof of Maurey's theorem depends on martingale inequalities, while our approach for general k is different. In fact, the bound we obtain for $k = 0$ is slightly better than (7) when γ is small.

Write $E = \cup_{i=1}^n (\{i\} \times A_i)$. Then

$$g_E(\pi) = |\{1 \le i \le n : \pi(i) \in A_i\}|.$$

For $0 \le m \le n$, let

$$S(E, F, m) = \{\pi \in \mathbb{S}_n : g_E(\pi) = m \ , \ \pi(i) \notin B_i \text{ for all } 1 \le i \le n\}.$$

Let

$$\ell(E) = \max_{1 \le i \le n} |A_i|. \tag{8}$$

10

Proposition 5. *Let E, F be as above. If $\ell(E) \leq \frac{n}{2}$, then*

$$|S(E,F)| = \sum_{m \leq \frac{\gamma n}{10}} |S(E,F,m)| \leq 4n^2 \left(\frac{n-k}{e}\right)^n e^{-\frac{\gamma n}{20}}.$$

Proof: For $1 \leq i \leq n$, let $R_i = A_i - B_i$, $S_i = [n] - A_i - B_i$ and denote $r_i = |R_i|$ and $p_i = \frac{r_i}{n-k} \leq \frac{2}{3}$ (as $r_i \leq \frac{n}{2}$ and $k \leq \frac{n}{4}$). Then $s_i = |S_i| = n - k - r_i$ and $\frac{s_i}{n-k} = 1 - p_i$. For a subset $I \subset [n]$ let M_I be the $n \times n$ matrix given by

$$M_I(i,j) = \begin{cases} 1 & i \in I, j \in R_i, \\ 1 & i \notin I, j \in S_i, \\ 0 & otherwise. \end{cases}$$

Note that $\pi \in S(E,F,m)$ iff there exists an $I \in \binom{[n]}{m}$ such that $\pi(i) \in R_i$ for $i \in I$ and $\pi(i) \in S_i$ for $i \in [n] - I$. The last condition is equivalent to $\prod_{i=1}^{n} M_I(i, \pi(i)) = 1$. As such I must be unique, it follows that

$$|S(E,F,m)| = \sum_{|I|=m} \sum_{\pi \in \mathbb{S}_n} \prod_{i=1}^{n} M_I(i, \pi(i)) = \sum_{|I|=m} \operatorname{per} M_I. \tag{9}$$

Since $s_i \geq n - |A_i| - k \geq \frac{n}{4}$, the Stirling approximation implies

$$s_i!^{\frac{1}{s_i}} \leq (e^2 s_i)^{\frac{1}{2s_i}} \left(\frac{s_i}{e}\right) \leq (2n)^{\frac{2}{n}} \left(\frac{s_i}{e}\right).$$

Recall the following result of Brégman (see e.g. Theorem 11.5 in [13]).

Theorem 6 (Brégman). *Let M be an $n \times n$ zero-one matrix with t_i ones in the i-th row. Then*

$$\operatorname{per} M \leq \prod_{i=1}^{n} t_i!^{\frac{1}{t_i}}.$$

Using (9) and Brégman's bound it follows that

$$
\begin{aligned}
|S(E, F, m)| &= \sum_{|I|=m} \operatorname{per} M_I \\
&\leq \sum_{|I|=m} \prod_{i \in I} r_i!^{\frac{1}{r_i}} \prod_{i \notin I} s_i!^{\frac{1}{s_i}} \\
&\leq \sum_{|I|=m} \prod_{i \in I} r_i \prod_{i \notin I} \left((2n)^{\frac{2}{n}} \left(\frac{s_i}{e} \right) \right) \\
&= (2n)^{\frac{2(n-m)}{n}} e^{m-n} (n-k)^n \sum_{|I|=m} \prod_{i \in I} \left(\frac{r_i}{n-k} \right) \prod_{i \notin I} \left(\frac{s_i}{n-k} \right) \\
&\leq 4n^2 \cdot e^m \cdot \left(\frac{n-k}{e} \right)^n \sum_{|I|=m} \prod_{i \in I} p_i \prod_{i \notin I} (1-p_i).
\end{aligned}
$$

$$(10)$$

Consider a sequence $\{Y_i\}_{i=1}^n$ of independent Bernoulli random variables such that $\Pr[Y_i = 1] = p_i = 1 - \Pr[Y_i = 0]$ and let $Y = \sum_{i=1}^n Y_i$. Then by (10)

$$
|S(E, F, m)| \leq 4n^2 \cdot e^m \cdot \left(\frac{n-k}{e} \right)^n \Pr[Y = m]. \tag{11}
$$

Next note that

$$
\begin{aligned}
E[Y] &= \sum_{i=1}^n p_i = \sum_{i=1}^n \frac{r_i}{n-k} \\
&\geq \frac{1}{n-k} \sum_{i=1}^n (|A_i| - k) \geq \frac{\gamma n^2 - kn}{n} \geq \frac{\gamma n}{2}.
\end{aligned}
$$

The Chernoff bound (see Theorem A.1.13 in [1]) states that for all $a > 0$

$$
\Pr[Y < E[Y] - a] < e^{-\frac{a^2}{2E[Y]}}. \tag{12}
$$

Using (11) and (12) with $a = \frac{4E[Y]}{5}$ it follows that

$$
\begin{aligned}
(4n^2)^{-1} \cdot e^{-\frac{\gamma n}{10}} &\cdot \left(\frac{n-k}{e} \right)^{-n} \sum_{m < \frac{\gamma n}{10}} |S(E, F, m)| \\
&\leq \Pr\left[Y < \frac{\gamma n}{10} \right] \leq \Pr\left[Y < \frac{E[Y]}{5} \right] \leq e^{-\frac{4\gamma n}{25}}
\end{aligned}
$$

$$(13)$$

12

and therefore

$$|S(E,F)| = \sum_{m < \frac{\gamma n}{10}} |S(E,F,m)| \leq 4n^2 \cdot \left(\frac{n-k}{e}\right)^n e^{-\frac{\gamma n}{20}}.$$

$\square$

Now let $E_1, \ldots, E_n$ be subsets of $[n]^2$ such that $\ell(E_i) \leq \frac{n}{2}$ for all $1 \leq i \leq n$. Suppose $I \subset [n]$ satisfies $|I| = \frac{\gamma n}{2}$ and $|E_i| \geq \gamma n^2$ for all $i \in I$. We next use Proposition 5 to bound the number of Latin squares $L = (\pi_1, \ldots, \pi_n)$ such that $g_{E_i}(\pi_i)$ is much smaller than its expected value for all $i \in I$. Let

$$\mathcal{G}(I) = \{L = (\pi_1, \ldots, \pi_n) \in \mathcal{L}_n : g_{E_i}(\pi_i) < \frac{\gamma n}{10} \text{ for all } i \in I\}.$$

The main ingredient in the proof of the large deviation bound for random Latin squares in Section 4.2 is the following

Proposition 7.

$$|\mathcal{G}(I)| \leq (2n)^{\gamma n} \left(\prod_{k=1}^{n} k!^{\frac{n}{k}}\right) e^{-\frac{\gamma^2 n^2}{40}}.$$

Proof: Let $k_0 = \frac{\gamma n}{2}$. Without loss of generality we may assume that $I = [k_0]$. A legal k-tuple $(\pi_1, \ldots, \pi_k) \in \mathbb{S}_n^k$ is *extendible to* $\mathcal{G}(I)$ if there exist $\pi_{k+1}, \ldots, \pi_n \in \mathbb{S}_n$ such that $(\pi_1, \ldots, \pi_n) \in \mathcal{G}(I)$. Fixing a $0 \leq k \leq n-1$ and a legal $(\pi_1, \ldots, \pi_k) \in \mathbb{S}_n^k$, we next obtain an upper bound on

$$\mu(\pi_1, \ldots, \pi_k) = |\{\pi_{k+1} \in S_k : (\pi_1, \ldots, \pi_{k+1}) \text{ is extendible to } \mathcal{G}(I)\}|.$$

For $1 \leq i \leq n$, let $B_i = \{\pi_j(i) : 1 \leq j \leq k\}$ and let $F = (B_1, \ldots, B_n)$. If $k < k_0$ then by Proposition 5

$$\mu(\pi_1, \ldots, \pi_k) \leq \sum_{m < \frac{\gamma n}{10}} |S(E_{k+1}, F, m)|$$
$$\leq 4n^2 \left(\frac{n-k}{e}\right)^n e^{-\frac{\gamma n}{20}}. \tag{14}$$

Suppose on the other hand that $k_0 \leq k < n$. If $(\pi_1, \ldots, \pi_{k+1})$ is extendible to $\mathcal{G}(I)$ then, in particular, it is legal and therefore $\pi_{k+1}(i) \notin B_i$ for all $1 \leq i \leq n$. Hence again by Brégman's bound

$$\mu(\pi_1, \ldots, \pi_k) \leq (n-k)!^{\frac{n}{n-k}}. \tag{15}$$

13

Choosing $\pi_1, \ldots, \pi_n$ sequentially and using (14) and (15) it follows that

$$|\mathcal{G}(I)| \leq \prod_{k=0}^{k_0-1} \left(4n^2 \left(\frac{n-k}{e} \right)^n e^{-\frac{\gamma n}{20}} \right) \prod_{k=k_0}^{n-1} (n-k)!^{\frac{n}{n-k}}$$

$$\leq (4n^2 e^{-\frac{\gamma n}{20}})^{\frac{\gamma n}{2}} \prod_{k=0}^{k_0-1} (n-k)!^{\frac{n}{n-k}} \prod_{k=k_0}^{n-1} (n-k)!^{\frac{n}{n-k}}$$

$$= (2n)^{\gamma n} \left(\prod_{k=1}^{n} k!^{\frac{n}{k}} \right) e^{-\frac{\gamma^2 n^2}{40}}.$$

$\square$

4.2 Proof of Theorem 4

Recall that $\mathcal{E}$ is a subset of 2-simplices of T_n of cardinality $|\mathcal{E}| \geq cn^3$. We have to bound the probability of

$$\mathcal{L}_n(c) = \{L \in \mathcal{L}_n : f_{\mathcal{E}}(L) < 10^{-3} c^2 n^2\}.$$

For $1 \leq k \leq n$, let

$$E'_k = \{(i,j) : [a_k, b_i, c_j] \in \mathcal{E}\} \subset [n]^2$$

and write $E'_k = \cup_{i=1}^{n} (\{i\} \times J'_{ki})$. For $1 \leq k, i \leq n$ choose an arbitrary $J_{ki} \subset J'_{ki}$ such that $|J_{ki}| = \min\{\frac{n}{2}, |J'_{ki}|\}$. Then $E_k = \cup_{i=1}^{n} (\{i\} \times J_{ki}) \subset E'_k$ satisfies $\ell(E_k) = \max_{1 \leq i \leq n} |J_{ki}| \leq \frac{n}{2}$ and $|E_k| \geq \frac{|E'_k|}{2}$. For $L = (\pi_1, \ldots, \pi_n) \in \mathcal{L}_n$, let

$$g(L) = \sum_{k=1}^{n} g_{E_k}(\pi_k).$$

Let $\gamma = \frac{c}{4}$. Then

$$E[g] = \frac{1}{n} \sum_{k=1}^{n} |E_k| \geq \frac{1}{2n} \sum_{k=1}^{n} |E'_k|$$

$$= \frac{|\mathcal{E}|}{2n} \geq \frac{cn^2}{2} = 2\gamma n^2.$$

Claim 8. *For any $L = (\pi_1, \ldots, \pi_n) \in \mathcal{L}_n(c)$, there exists a subset $I_L \subset [n]$ of size $|I_L| = \frac{\gamma n}{2}$ such that for all $i \in I_L$ both $|E_i| \geq \gamma n^2$ and $g_{E_i}(\pi_i) \leq \frac{\gamma n}{10}$.*

14

Proof: Let
$$I = \{1 \le i \le n : |E_i| \ge \gamma n^2\}$$

and
$$J = \{i \in I : g_{E_i}(\pi_i) \le \frac{\gamma n}{10}\}.$$

Since $\sum_{k=1}^{n} |E_k| \ge \frac{1}{2}\sum_{k=1}^{n} |E_k'| = \frac{|\mathcal{E}|}{2} \ge 2\gamma n^3$ it follows that $|I| \ge \gamma n$. Therefore

$$(\gamma n - |J|)\frac{\gamma n}{10} \le (|I| - |J|)\frac{\gamma n}{10} \le \sum_{i=1}^{n} g_{E_i}(\pi_i)$$

$$\le \sum_{i=1}^{n} g_{E_i'}(\pi_i) = f_{\mathcal{E}}(L) < 10^{-3}c^2 n^2 = 16 \cdot 10^{-3}\gamma^2 n^2.$$

Hence $|J| \ge (1 - \frac{16}{100})\gamma n > \frac{\gamma n}{2}$. The Claim follows by taking I_L to be any $\frac{\gamma n}{2}$ subset of J.

$\square$

Claim 8 implies that $\mathcal{L}_n(c) \subset \bigcup_{|I|=\frac{\gamma n}{2}} \mathcal{G}(I)$. Therefore by Proposition 7

$$|\mathcal{L}_n(c)| \le |\bigcup_{|I|=\frac{\gamma n}{2}} \mathcal{G}(I)|$$

$$\le \binom{n}{\frac{\gamma n}{2}}(2n)^{\gamma n}\left(\prod_{k=1}^{n} k!^{\frac{n}{k}}\right) e^{-\frac{\gamma^2 n^2}{40}} \tag{16}$$

$$= \binom{n}{\frac{cn}{8}}(2n)^{\frac{cn}{4}}\left(\prod_{k=1}^{n} k!^{\frac{n}{k}}\right) e^{-\frac{c^2 n^2}{640}}.$$

By a classical result on the enumeration of Latin squares (See Theorem 17.3 in [13])

$$1 \le \frac{\prod_{k=1}^{n} k!^{\frac{n}{k}}}{|\mathcal{L}_n|} = (1 + o(1))^{n^2}. \tag{17}$$

It follows by (16) that for sufficiently large $n \ge n_0(c)$

$$\Pr[\mathcal{L}_n(c)] = \frac{|\mathcal{L}_n(c)|}{|\mathcal{L}_n|} \le e^{-10^{-3}c^2 n^2}.$$

$\square$

5   Expanders from Latin Squares

Here we prove Theorem 1. We first consider the expansion of small cochains. For a d-tuple of permutations $\tilde{\pi} = (\pi_1, \ldots, \pi_d) \in \mathbb{S}_n^d$, let $G = G(\tilde{\pi})$ be the n by n bipartite graph whose edge set is

$$\{(i, \pi_j(i)) : 1 \leq i \leq n, 1 \leq j \leq d\} \subset [n]^2.$$

Let $\mathcal{G}(n, d)$ be the uniform probability space $\{G(\tilde{\pi}) : \tilde{\pi} \in \mathbb{S}_n^d\}$. Friedman's theorem [4, 5] on the spectral gap of d-regular graphs implies that if $d \geq 100$ is fixed then $\mu(G) > d - 3\sqrt{d}$ with with probability $1 - O(n^{-2})$.

Consider a random $Y(\underline{L}^d) \in \mathcal{Y}(n, d)$. The link $\mathrm{lk}(Y(\underline{L}^d), v)$ of a fixed vertex $v \in V$ is a random graph in $\mathcal{G}(n, d)$ hence

$$\Pr\left[\mu(\mathrm{lk}(Y(\underline{L}^d), v)) > d - 3\sqrt{d}\right] = 1 - O(n^{-2}).$$

Since $|V| = 3n$ it follows that

$$\Pr\left[\min_{v \in V} \mu(\mathrm{lk}(Y(\underline{L}^d), v)) > d - 3\sqrt{d}\right] = 1 - O(n^{-1}).$$

Proposition 2 thus implies the following

Corollary 9. *For any fixed $d \geq 100$ and $c < 1$, the following holds with probability $1 - O(n^{-1})$:*

$$\frac{\|d_1\phi\|_{Y(\underline{L}^d)}}{\|[\phi]\|} \geq \frac{(d - 3\sqrt{d})(1 - c^{1/3})}{2} - \frac{d}{3}$$

for all 1-cochains $\phi \in C^1(T_n; \mathbb{F}_2)$ such that $\|[\phi]\| \leq cn^2$.

$\square$

We next consider the expansion of large cochains. Fix $\phi \in C^1(T_n; \mathbb{F}_2)$ such that $\|[\phi]\| \geq cn^2$ with $c < 1$. A special case of a result of Gromov (Section 2.13 in [7]) asserts that $h_1(T_n) = \Omega(n)$. The constant in Gromov's bound was improved by Dotterer and Kahle (Proposition 5.7 in [2]) who showed that $h_1(T_n) \geq \frac{n}{5}$. It follows that

$$\mathcal{E} = \{\sigma \in T_n(2) : d_1\phi(\sigma) \neq 0\}$$

16

satisfies

$$|\mathcal{E}| = \|d_1\phi\|_{T_n} \geq \frac{n}{5}\|[\phi]\| \geq \frac{cn^3}{5}.$$

If $L \in \mathcal{L}_n$ then $\|d_1\phi\|_{Y(L)} = |Y(L)(2) \cap \mathcal{E}| = f_{\mathcal{E}}(L)$. Theorem 4 then implies that if $n \geq n_0(c/5)$ then

$$\begin{aligned}
\Pr\left[L \in \mathcal{L}_n : \|d_1\phi\|_{Y(L)} < 10^{-3}(c/5)^2 n^2\right] \\
= \Pr\left[L \in \mathcal{L}_n : f_{\mathcal{E}}(L) < 10^{-3}(c/5)^2 n^2\right] \\
< e^{-10^{-3}(c/5)^2 n^2} = e^{-4 \cdot 10^{-5} c^2 n^2}.
\end{aligned} \tag{18}$$

Let $\underline{L}^d = (L_1, \ldots, L_d) \in \mathcal{L}_n^d$. Then

$$\|d_1\phi\|_{Y(\underline{L}^d)} \geq \max_{1 \leq i \leq d} \|d_1\phi\|_{Y(L_i)}.$$

Therefore by (18)

$$\Pr\left[\underline{L}^d \in \mathcal{L}_n^d : \|d_1\phi\|_{Y(\underline{L}^d)} < 4 \cdot 10^{-5} c^2 n^2\right] < e^{-4 \cdot 10^{-5} dc^2 n^2}.$$

As the number of 1-cochains is 2^{3n^2} and the support of a 1-cochain is at most $3n^2$ we obtain

Corollary 10. *If $n \geq n_0(c/5)$ then the following holds with probability at least $1 - 2^{3n^2} e^{-4 \cdot 10^{-5} dc^2 n^2}$:*

$$\frac{\|d_1\phi\|_{Y(\underline{L}^d)}}{\|[\phi]\|} \geq 10^{-5} c^2$$

for all 1-cochains $\phi \in C^1(T_n; \mathbb{F}_2)$ such that $\|\phi\| \geq cn^2$.

$\square$

Proof of Theorem 1: Let $c = 10^{-3}$ then

$$\frac{(d - 3\sqrt{d})(1 - c^{1/3})}{2} - \frac{d}{3} > 1$$

for $d > 200$. Let $d = 10^{11}$ then $2^3 < e^{4 \cdot 10^{-5} dc^2}$ so Corollaries 9 and 10 imply that Theorem 1 holds with $d = 10^{11}$ and $\epsilon = 10^{-5} c^2 = 10^{-11}$.

$\square$

6   Concluding Remarks

We have shown that there exist fixed $d < \infty$ and $\epsilon > 0$ such that $Y \in \mathcal{Y}(n,d)$ a.a.s. satisfies $h_1(Y) > \epsilon$. In particular, there exist infinite families of $(2, d, \epsilon)$-expanders. We conclude with the following comments and questions.

1. Let d_0 be the smallest d for which Theorem 1 remains true. It can be shown that if $d = 3$ then

$$\lim_{n \to \infty} \Pr\left[H_1(Y(L_1, L_2, L_3); \mathbb{F}_2) \neq 0\right] \geq 1 - \frac{17e^{-3}}{2} \doteq 0.57.$$

   In particular $d_0 \geq 4$. It seems plausible that d_0 is in fact 4. A considerably weaker question would be to determine for a fixed field $\mathbb{F}$ the minimal d such that $H_1(Y; \mathbb{F}) = 0$ a.a.s. for $Y \in \mathcal{Y}(n,d)$. For $\mathbb{F} = \mathbb{R}$ one can use a spectral approach similar to the one applied in [8] and [9] for other models. A classical result of Garland [6] asserts (roughly) that if in a 2-dimensional complex Y all vertex links have sufficiently large spectral gaps then $H_1(Y; \mathbb{R}) = 0$. Combining Friedman's eigenvalue bounds for random graphs [4, 5] and Garland's theorem it can be shown that if $d \geq 100$ then $H_1(Y; \mathbb{R}) = 0$ a.a.s. for $Y \in \mathcal{Y}(n,d)$.

2. The complexes $Y \in \mathcal{Y}(n,d)$ satisfy $D_1(Y) \leq d$ but $D_0(Y) = 2n$. It would be very interesting to prove the existence (or even better to give explicit constructions) of infinite families of ϵ-expanding complexes X such that both $D_0(X)$ and $D_1(X)$ are bounded. For some results in this direction see the work of Kaufman, Kazhdan and Lubotzky [11] on expansion in Ramanujan complexes.

3. The model $\mathcal{Y}(n,d)$ generalizes in a straightforward manner to higher dimensions and it seems likely that Theorem 1 remains true there. The main obstacle to extending the present proof to this case is the absence (at present) of higher dimensional analogues of the asymptotic enumeration formula (17).

4. A simplicial complex X is said to have the (k, δ) topological overlap property if for any continuous map $f : X \to \mathbb{R}^k$ there exists a point $p \in \mathbb{R}^k$ such that

$$|\{\sigma \in X(k) : p \in f(\sigma)\}| \geq \delta|X(k)|.$$

A remarkable theorem of Gromov [7] asserts that for any k and $\epsilon > 0$ there exists an $\delta = \delta(k, \epsilon)$ such that if $h_i(X) \geq \epsilon \cdot \frac{|X(i+1)|}{|X(i)|}$ for every $0 \leq i \leq k-1$, then X has the (k, δ) topological overlap property. Theorem 1 therefore implies that there exist d and $\delta > 0$ such that $Y \in Y(n, d)$ a.a.s. has the $(2, \delta)$ topological overlap property.

References

[1] N. Alon and J. Spencer, *The Probabilistic Method*, 3rd Edition, Wiley-Intescience, 2008.

[2] D. Dotterrer and M. Kahle, Coboundary expanders, *J. Topol. Anal.* **4**(2012) 499-514.

[3] J. Fox, M. Gromov, V. Lafforgue, A. Naor and J. Pach, Overlap properties of geometric expanders, *J. Reine Angew. Math.* **671**(2012) 49-83.

[4] J. Friedman, On the second eigenvalue and random walks in random d-regular graphs, *Combinatorica* **11** (1991) 331-362.

[5] J. Friedman, A proof of Alon's second eigenvalue conjecture and related problems. Mem. Amer. Math. Soc. **195**(2008).

[6] H. Garland, *p*-adic curvature and the cohomology of discrete subgroups of *p*-adic groups, *Annals of Math.* **97** (1973) 375-423.

[7] M. Gromov, Singularities, expanders and topology of maps. Part 2: From combinatorics to topology via algebraic isoperimetry, *Geom. Funct. Anal.* **20**(2010) 416-526.

[8] A. Gundert and U. Wagner, On Laplacians of Random Complexes, Proc. 28th Annual ACM Symposium on Computational Geometry (2012) 151–160.

[9] C. Hoffman, M. Kahle and E. Paquette, A sharp threshold for Kazhdan's property (T), arXiv:1201.0425

[10] S. Hoory, N. Linial and A. Wigderson, Expander graphs and their applications, *Bull. Amer. Math. Soc. (N.S.)* **43**(2006) 439-561.

[11] T. Kaufman, D. Kazhdan and A. Lubotzky, Isoperimetric inequalities for Ramanujan complexes, in preparation.

[12] N. Linial and R. Meshulam, Homological connectivity of random 2-complexes, *Combinatorica* **26**(2006) 475–487.

[13] J. H. van Lint and R. M. Wilson, A Course in Combinatorics. Second edition. Cambridge University Press, Cambridge, 2001.

[14] A. Lubotzky, Discrete Groups, Expanding Graphs and Invariant Measures. With an appendix by Jonathan D. Rogawski. Progress in Mathematics, 125. Birkhäuser Verlag, Basel, 1994.

[15] A. Lubotzky, Expander graphs in pure and applied mathematics, *Bull. Amer. Math. Soc. (N.S.)* **49**(2012) 113-162.

[16] A. Lubotzky, Ramanujan complexes and high dimensional expanders, Takagi Lectures, Tokyo, 2012, arXiv:1301.1028.

[17] A. Lubotzky, B. Samuels and U. Vishne, Ramanujan complexes of type A_d, *Israel J. Math.* **149**(2005) 267-299.

[18] B. Maurey, Construction de suites symétriques, C. R. Acad. Sci. Paris Sér. A-B **288**(1979), no. 14, A679-A681.

[19] R. Meshulam and N. Wallach, Homological connectivity of random k-dimensional complexes, *Random Struct. Algorithms* **34**(2009) 408–417.

[20] V. D. Milman, G. Schechtman, Asymptotic Theory of Finite-Dimensional Normed Spaces. With an appendix by M. Gromov. Lecture Notes in Mathematics, 1200. Springer-Verlag, Berlin, 1986.