On the trace of incompatible vectors

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Abstract

Let $A = \{a_1, \ldots, a_t\} \subset \{0,1\}^n$, $F_1, \ldots, F_t \subset [n] = \{1, \ldots, n\}$. The ordered family of pairs $((a_1, F_1), \ldots, (a_t, F_t))$ is *incompatible* if for any $1 \le k < l \le t$ there exists a $j \in F_k$ such that $a_{kj} \neq a_{lj}$.
It is shown that for any incompatible family $((a_1, F_1), \ldots, (a_t, F_t))$, there exists a $1-1$ mapping $\phi : [t] \to 2^{[n]}$ such that $\phi(i) \subset F_i$ and $\phi(i)$ is shattered by A.

1 On the trace of incompatible vectors

Let $a = (a_1, \ldots, a_n)$ be a vector in $\{0,1\}^n$, and S a subset of $[n] = \{1, \ldots, n\}$. The *trace* of a on S is $a_{|S} = (a_j \ : \ j \in S) \in \{0,1\}^S$.
A subset $S \subset [n] = \{1, \ldots, n\}$ is *shattered* by $A \subset \{0,1\}^n$ if for each $\epsilon = (\epsilon_j \ : \ j \in S) \in \{0,1\}^S$ there exists an $a \in A$ such that $a_{|S} = \epsilon$.
Let $\mathcal{S}(A)$ denote the family of all $S \subset [n]$ which are shattered by A.
The following basic result was proved by Sauer [5] and Perles and Shelah [6].

Theorem 1.1 ([5],[6]) *If $|A| > \sum_{i=0}^{d} \binom{n}{i}$ then there exists an $S \in \mathcal{S}(A)$ such that $|S| > d$.*

Different proofs and extensions of Theorem 1.1 were given by Alon [1], Frankl [2], Frankl and Pach [3] and others. See the recent survey by Füredi and Pach [4].
We first give a simple proof of the following extension of Theorem 1.1 which is also implicit in all previous proofs.

Theorem 1.2 $|A| \le |\mathcal{S}(A)|$ *for any* $A \subset \{0,1\}^n$

Proof: Let U denote the $\mathbb{Z}_2$-linear space of multilinear polynomials in $\mathbb{Z}_2[x_1, \ldots, x_n]$. With each $a = (a_1, \ldots, a_n) \in A$ we associate the polynomial $f_a(x) = \prod_{j=1}^{n}(x_j + a_j + 1) \in U$. For each non-shattered subset $T \in 2^{[n]} - \mathcal{S}(A)$ we choose a vector $b_T = (b_{T,j} \ : \ j \in T) \in \{0,1\}^T$ such that $b_T \neq a_{|T}$ for all $a \in A$. Let $g_T(x) = \prod_{j \in T}(x_j + b_j + 1) \in U$. Note that for $a, a' \in A$, $f_a(a') = \delta(a, a')$, and $g_T(a) = 0$ for all $T \notin \mathcal{S}(A)$.

Claim 1.3 *The family*

$$\{f_a(x) \ : \ a \in A\} \cup \{g_T(x) \ : \ T \notin \mathcal{S}(A)\}$$

is linearly independent in U.

Proof: Suppose

$$\sum_{a \in A} \alpha_a f_a(x) + \sum_{T \notin \mathcal{S}(A)} \beta_T g_T(x) = 0. \tag{1}$$

Substituting $a' \in A$ in Eq. (1) we obtain $\alpha_{a'} = 0$. It thus remains to show that $\{g_T(x) \ : \ T \notin \mathcal{S}(A)\}$ is linearly independent. This follows from the fact that the unique highest degree monomials in the expansions of the g_T's are all different.

$\square$

Claim 1.3 implies $|A| + (2^n - |\mathcal{S}(A)|) \le \dim U = 2^n$, hence $|A| \le |\mathcal{S}(A)|$.

$\square$

Next we consider the following extension of Theorem 1.2. Let $A = \{a_1, \ldots, a_t\} \subset \{0,1\}^n$, where $a_i = (a_{i1}, \ldots, a_{in})$, and let $F_1, \ldots, F_t \subset [n]$. The ordered family of pairs $((a_1, F_1), \ldots, (a_t, F_t))$ is *incompatible* if for any $1 \le k < l \le t$ there exists a $j \in F_k$ such that $a_{kj} \neq a_{lj}$.

Theorem 1.4 *For any incompatible family* $((a_1, F_1), \ldots, (a_t, F_t))$*, there exists a* $1-1$ *mapping* $\phi : [t] \to 2^{[n]}$ *such that* $\phi(i) \subset F_i$ *and* $\phi(i)$ *is shattered by* A.

Proof: With each pair (a_i, F_i) we associate the polynomial $f_i(x) = \prod_{j \in F_i}(x_j + a_{ij} + 1) \in U$. Let $V = \{g(x) \in U : g(a_i) = 0 \text{ for all } 1 \leq i \leq t\}$, and let $W = U/V$. For $f \in U$ let $\overline{f} \in W$ denote the image of f under the quotient map.

Claim 1.5 *$\overline{f_1}, \ldots, \overline{f_t}$ are linearly independent in W.*

Proof: Suppose $\sum_{k=1}^{t} \lambda_k f_k(x) \in V$. The incompatibility condition implies that $f_k(a_l) = 0$ whenever $k < l$. It follows that for all $1 \leq l \leq t$

$$0 = \sum_{k=1}^{t} \lambda_k f_k(a_l) = \lambda_l + \sum_{k=l+1}^{t} \lambda_k f_k(a_l)$$

and so $\lambda_1 = \cdots = \lambda_t = 0$.

$\square$

Claim 1.6 *For any $F \subset [n]$*

$$\overline{\prod_{j \in F} x_j} \in \text{Span } \{\overline{\prod_{j \in S} x_j} \ : \ S \in 2^F \cap \mathcal{S}(A)\}.$$

Proof: We apply induction on $|F|$. If $F \in \mathcal{S}(A)$ then we are done. Otherwise there exists an $\epsilon \in \{0,1\}^F$ such that $\epsilon \neq a_{i|F}$ for all $1 \leq i \leq t$.
It follows that $g(x) = \prod_{j \in F}(x_j + \epsilon_j + 1)$ satisfies $g(a_i) = 0$ for all $1 \leq i \leq t$ and so $g(x) \in V$. Therefore

$$\overline{\prod_{j \in F} x_j} = \overline{\prod_{j \in F} x_j - g(x)} \in \text{Span } \{\overline{\prod_{j \in F'} x_j} \ : \ F' \subsetneqq F\}$$

and the Claim follows from the induction hypothesis.

$\square$

Claim 1.6 implies that for each $1 \leq i \leq t$ we may expand

$$\overline{f_i(x)} = \sum_{S \in 2^{F_i} \cap \mathcal{S}(A)} \mu_{i,S} \overline{\prod_{j \in S} x_j}.$$

Consider the $t \times 2^n$ matrix M indexed by $[t] \times 2^{[n]}$ and given by $M(i,S) = \mu_{i,S}$ if $S \in 2^{F_i} \cap \mathcal{S}(A)$, and zero otherwise.

Claim 1.5 implies that $\text{rank}_{\mathbb{Z}_2} M = t$, so in particular there exists a $1-1$ mapping $\phi : [t] \to 2^{[n]}$ such that $M(i, \phi(i)) \neq 0$ for all $1 \leq i \leq t$. It follows that $\phi(i) \subset F_i$ and that $\phi(i)$ is shattered by A.

$\square$

For a vector $a \in \{0,1\}^n$ let Supp $a = \{1 \leq i \leq n \ : \ a_i = 1\}$.
Let $A \subset \{0,1\}^n$ and let $a_1, \ldots, a_t$ be an ordering of A such that
$|\text{Supp } a_k| \geq |\text{Supp } a_l|$ for all $k \leq l$.
The ordered family of pairs $((a_1, \text{Supp } a_1), \ldots, (a_t, \text{Supp } a_t))$ is clearly incompatible, hence Theorem 1.4 implies the following result implicit in Frankl and Pach [3]:

Corollary 1.7 ([3]) *For any $A \subset \{0,1\}^n$ there exists a $1-1$ mapping $\phi : A \to \mathcal{S}(A)$ such that $\phi(a) \subset$ Supp a for all $a \in A$.*

$\square$

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References

[1] N. Alon, On the density of sets of vectors, Discrete Mathematics, 46(1983) 199-202.

[2] P. Frankl, On the traces of finite sets, Journal of Combinatorial Theory Ser. A, 34(1983) 41-45.

[3] P. Frankl and J. Pach, On the number of sets in a null $t-$design, European Journal of Combinatorics, 4(1983) 21-23.

[4] Z. Füredi and J. Pach, Traces of finite sets: Extremal problems and geometric applications, in: *Extremal Problems for Finite Sets*, Bolayi Society Mathematical Studies 3, 1991, Edited by: P. Frankl, Z. Füredi, G. Katona and D. Miklós, pp. 251-282.

[5] N. Sauer, On the density of families of sets, Journal of Combinatorial Theory Ser.A, 13(1972) 145-147.

[6] S. Shelah, A combinatorial problem: stability and order for models and theories in infinitary languages, Pacific J. Math., 41(1972) 247-261.