NORTH-HOLLAND

On Two-Parameter Families of Symmetric Matrices

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ABSTRACT

Let f be a homogeneous polynomial mapping of degree d from $\mathbf{C}^2$ into the space of complex $n \times n$ symmetric matrices. It is shown that if $\operatorname{rank} f(x) = k$ for all $0 \neq x \in \mathbf{C}^2$, then either k is even or d is even.

1. INTRODUCTION

Let $M_{m\times n}(\mathbf{C})$ denote the space of complex $m \times n$ matrices. A mapping $f : \mathbf{C}^t \to M_{m\times n}(\mathbf{C})$ is a *k-mapping* if $\operatorname{rank} f(x) = k$ for all $0 \neq x \in \mathbf{C}^t$. The mapping $f = (f_{ij})_{i=1}^{m}{}_{j=1}^{n}$ is *d-homogeneous* if each of its components f_{ij} is a homogeneous polynomial of degree d.

Let $S_n(\mathbf{C})$ denote the space of complex $n \times n$ symmetric matrices. Here we show:

THEOREM 1.1. *If $f : \mathbf{C}^2 \to S_n(\mathbf{C})$ is a d-homogeneous k-mapping, then either d is even or k is even.*

The following corollary was proved by R. Westwick (Theorem 3, in [1]), and independently by A. Malek [5]. R. Loewy had proved the corollary for spaces of real symmetric matrices [5].

COROLLARY 1.2. *Let k be odd. If W is a linear subspace of $S_n(\mathbf{C})$ such that* $\operatorname{rank} A = k$ *for all* $0 \neq A \in W$, *then* $\dim W \leq 1$. ∎

Theorem 1.1 is proved in Section 2 using a theorem of Grothendieck on holomorphic vector bundles over the complex projective line.

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The following examples show that if either d or k is even and n is sufficiently large, then there exist d-homogeneous k-mappings $f : \mathbf{C}^2 \to S_n(\mathbf{C})$:

(1) $k = 2k'$. For $n \geq k + 1$ define $f : \mathbf{C}^2 \to S_n(\mathbf{C})$ by $f_{i,\, i+k'}(s, t) = f_{i+k',\, i}(s, t) = s^d$ and $f_{i,\, i+k'+1}(s, t) = f_{i+k'+1,\, i}(s, t) = t^d$ for $1 \leq i \leq k'$, and $f_{ij}(s, t) = 0$ otherwise.

(2) $d = 2d'$. For $n \geq 2k$ define $f : \mathbf{C}^2 \to S_n(\mathbf{C})$ by $f_{2i-1,\, 2i-1}(s, t) = s^d$, $f_{2i,\, 2i}(s, t) = t^d$, and $f_{2i-1,\, 2i}(s, t) = f_{2i,\, 2i-1}(s, t) = (st)^{d'}$ for $1 \leq i \leq k$, and $f_{ij}(s, t) = 0$ otherwise.

2. k-MAPPINGS AND VECTOR BUNDLES

Let $\mathbf{CP}^{t-1}$ denote the $t-1$-dimensional complex projective space, regarded as a quotient space $\mathbf{C}^t - \{0\}/\sim$, where $x \sim y$ iff $x = \lambda y$ for some $\lambda \in \mathbf{C}^*$. Let $[x] \in \mathbf{CP}^{t-1}$ denote the image of $x \in \mathbf{C}^t$ under the quotient map.

Let $\mathcal{O}(1)$ denote the hyperplane bundle over $\mathbf{CP}^{t-1}$, and let $\mathcal{O}(-1) = \mathrm{Hom}(\mathcal{O}(1), \mathbf{C})$ denote the tautological line bundle over $\mathbf{CP}^{t-1}$. For $l \geq 1$ let $\mathcal{O}(l) = \mathcal{O}(1)^{\otimes l}$ and $\mathcal{O}(-l) = \mathcal{O}(-1)^{\otimes l}$ (see [2 p. 145]).

The total space of $\mathcal{O}(-l)$ is given by

$$E(\mathcal{O}(-l)) = \{([x], v_1 \otimes \cdots \otimes v_l) : [x] \in \mathbf{CP}^{t-1},\ v_1, \ldots, v_l \in \mathrm{Span}\{x\}\}.$$

For a matrix $A \in M_{m \times n}(\mathbf{C})$ let $\mathrm{Row}\, A$ ($\mathrm{Col}\, A$) denote the space spanned by the rows (columns) of A.

Let $f : \mathbf{C}^t \to M_{m \times n}(\mathbf{C})$ be a d-homogeneous k-mapping. We associate with f the holomorphic k-plane bundle $\xi(f)$ whose total space is

$$E(\xi(f)) = \{([x], v) : [x] \in \mathbf{CP}^{t-1},\ v \in \mathrm{Row} f(x)\}.$$

Let $f^T : \mathbf{C}^t \to M_{n \times m}(\mathbf{C})$ denote the transpose of $f : f^T(x) = f(x)^T$. The total space of $\xi(f^T)$ is $\{([x], u) : [x] \in \mathbf{CP}^{t-1},\ u \in \mathrm{Col} f(x)\}$.

PROPOSITION 2.1. *Let $f : \mathbf{C}^t \to M_{m \times n}(\mathbf{C})$ be a d-homogeneous k-mapping, then*

$$\xi(f^T) \cong \mathrm{Hom}(\xi(f), \mathbf{C}) \otimes \mathcal{O}(-d).$$

PROOF. In the following we regard $f(x)$ both as an $m \times n$ matrix and as a linear transformation from $\mathbf{C}^n$ to $\mathbf{C}^m$.

The fiber of $\xi(f)$ over $[x] \in \mathbf{CP}^{t-1}$ is $[x] \times \mathrm{Row} f(x)$. Its dual may be identified with the quotient space $[x] \times \mathbf{C}^n/\ker f(x)$ as follows: For a column vector $z = (z_1, \ldots, z_n)^T \in \mathbf{C}^n$ let $\bar{z} = z + \ker f(x) \in \mathbf{C}^n/\ker f(x)$. For $v = (v, \ldots, v_n) \in$

Row$f(x)$ let $([x], \bar{z})$ act on $([x], v)$ by $v \cdot z = \sum_{i=1}^{n} v_i z_i$. Thus the total space of the dual bundle $\mathrm{Hom}(\xi(f), \mathbf{C})$ is

$$\{([x], \bar{z}) : [x] \in \mathbf{CP}^{t-1}, \bar{z} \in \mathbf{C}^n / \ker f(x)\},$$

and the total space of $\mathrm{Hom}(\xi(f), \mathbf{C}) \otimes \mathcal{O}(-d)$ is

$$\{([x], \bar{z} \otimes \lambda_1 x \otimes \cdots \otimes \lambda_d x) : [x] \in \mathbf{CP}^{t-1}, \bar{z} \in \mathbf{C}^n / \ker f(x), \lambda_1, \ldots, \lambda_d \in \mathbf{C}\}.$$

We now define a vector bundle morphism

$$\varphi : \mathrm{Hom}(\xi(f), \mathbf{C}) \otimes \mathcal{O}(-d) \to \xi(f^T)$$

by

$$\varphi([x], \bar{z} \otimes \lambda_1 x \otimes \cdots \otimes \lambda_d x) = ([x], \lambda_1 \cdots \lambda_d f(x) z).$$

The d-homogeneity of f implies that φ is a well-defined isomorphism. ■

We shall need the following result of Grothendieck [3]:

THEOREM 2.2 (Grothendieck). *For any holomorphic k-plane bundle ξ over $\mathbf{CP}^1$ there exists a unique sequence of integers $a_1 \geq \cdots \geq a_k$ such that $\xi \cong \bigoplus_{i=1}^{k} \mathcal{O}(a_i)$.*

Proof of Theorem 1.1. Let $f : \mathbf{C}^2 \to S_n(\mathbf{C})$ be a d-homogeneous k-mapping, and consider $\xi(f)$, the holomorphic k-plane bundle associated with f. By Grothendieck's theorem $\xi(f) \cong \bigoplus_{i=1}^{k} \mathcal{O}(a_i)$ for some $a_1 \geq \cdots \geq a_k$; hence by Proposition 2.1,

$$\bigoplus_{i=1}^{k} \mathcal{O}(a_i) \cong \xi(f) = \xi(f^T) \cong \mathrm{Hom}(\xi(f), \mathbf{C}) \otimes \mathcal{O}(-d) \cong \bigoplus_{i=1}^{k} \mathcal{O}(-a_i - d).$$

It follows that $a_i = -a_{k+1-i} - d$ for all $1 \leq i \leq k$. Thus if k is odd, then $d = -2a_{k+1/2}$ is even. ■

REMARKS

(1) It is shown in Hartshorne [4, p. 384, Exercise 2.6] that Grothendieck's theorem generalizes to any algebraically closed field (I am grateful to A. Yekutieli for this information). It follows that Theorem 1.1 is valid over any algebraically closed field.

(2) For other applications of vector bundles to spaces of matrices of fixed rank see [7, 8, 6].

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