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On *k*-Spaces of Real Matrices

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It is shown that if W is a linear subspace of real $n \times n$ matrices, such that $\operatorname{rank}(A) = k$ for all $0 \neq A \in W$, then $\dim W \leqslant n$. If $\dim W = n$, $5 \leqslant n$ is prime, and 2 is primitive modulo n, then $k = 1$.

1. INTRODUCTION

Let $M_n(\mathbf{R})$ denote the space of real $n \times n$ matrices. A linear subspace $W \subseteq M_n(\mathbf{R})$ is called a k-space, if $\operatorname{rank}(A) = k$ for all $0 \neq A \in W$. In this note we consider k-spaces of maximal dimension.

THEOREM 1 *The dimension of a k-space $W \subseteq M_n(\mathbf{R})$ is at most n.*

Next we consider an extremal case of Theorem 1. Let $f(n)$ denote the maximal k for which there exists an n-dimensional k-space. A lower bound on $f(n)$ may be obtained as follows:

Let $\rho(n)$ denote the maximal dimension of an n-space of $M_n(\mathbf{R})$. By the Radon–Hurwitz construction, and Adams vector field theorem (see [1]), $\rho(n) = 2^c + 8d$, where $n = (2a+1)2^{c+4d}$, a, c, d are integers, and $0 \leqslant c \leqslant 3$.

Suppose now that $A_1, \ldots, A_k$ form a basis of an n-space of $M_n(\mathbf{R})$. Using column notation, we define $W \subseteq M_n(\mathbf{R})$ as the collection of all matrices of the form $[A_1x; \ldots; A_kx; 0; \ldots; 0]$ where x ranges over $\mathbf{R}^n$.

Clearly W is an n-dimensional k-space of $M_n(\mathbf{R})$, and hence $f(n) \geqslant \rho(n) \geqslant 1$. Equality does not always hold: The skew-symmetric 3×3 matrices form a 3-dimensional 2-space, so $f(3) = 2 > 1 = \rho(3)$.

On the other hand we show:

THEOREM 2 *If $5 \leqslant n$ is a prime and 2 generates $\mathbf{Z}_n^*$, then $f(n) = 1$.*

The proofs are topological and are based on a construction of J. Sylvester [5] which associates a bundle map to a k-space of matrices.

2. *k*-SPACES AND BUNDLE MAPS

Let $\mathbf{P}^{m-1} = \mathbf{S}^{m-1}/\sim$ denote the $m-1$-dimensional real projective space, viewed as a quotient space of $\mathbf{S}^{m-1} \subseteq \mathbf{R}^m$, under the relation $x \sim -x$. We denote the equivalence class of $x \in \mathbf{S}^{m-1}$ by $[x] \in \mathbf{P}^{m-1}$.

Let γ_{m-1} denote the canonical line bundle over $\mathbf{P}^{m-1}$, and let $n\gamma_{m-1}$ be the n-fold Whitney sum $\gamma_{m-1} \oplus \cdots \oplus \gamma_{m-1}$. The elements of the total space of $n\gamma_{m-1}$ are of the form $([x], \lambda_1 x, \ldots, \lambda_n x)$ where $x \in \mathbf{S}^{m-1}$ and $\lambda_1, \ldots, \lambda_n \in \mathbf{R}$. Finally, let ε^n denote the trivial n-plane bundle over $\mathbf{P}^{m-1}$.

Suppose now that $W = \mathrm{Span}\{A_1, \ldots, A_m\}$ is an m-dimensional k-space of $M_n(\mathbf{R})$. As in Sylvester's paper [5], we define a bundle map $f: n\gamma_{m-1} \to \varepsilon^n$ by:

$$f([x], \lambda_1 x, \ldots, \lambda_n x) = \left([x], \left(\sum_{i=1}^{m} x_i A_i \right) \begin{pmatrix} \lambda_1 \\ \vdots \\ \lambda_n \end{pmatrix} \right)$$

where $x = (x_1, \ldots, x_m) \in \mathbf{S}^{m-1}$.

Since W is a k-space, it follows that f is a mapping of constant rank, hence the image and kernel bundles are defined and satisfy:

$$\mathrm{Im}\, f \oplus \ker f \cong n\gamma_{m-1},$$

$$\mathrm{Im}\, f \oplus (\mathrm{Im}\, f)^{\perp} \cong \varepsilon^n.$$

Applying the total Stiefel–Whitney classes (see [4]), we obtain the following equalities in $H^*(\mathbf{P}^{m-1}; \mathbf{Z}_2)$:

$$\omega(\mathrm{Im}\, f) \cdot \omega(\ker f) = \omega(n\gamma_{m-1}) = (1 + a)^n, \tag{1}$$

$$\omega(\mathrm{Im}\, f) \cdot \omega((\mathrm{Im}\, f)^{\perp}) = \omega(\varepsilon^n) = 1 \tag{2}$$

where $0 \neq a \in H^1(\mathbf{P}^{m-1}; \mathbf{Z}_2)$.

Taking the image of (1) and (2) under the isomorphism $H^*(\mathbf{P}^{m-1}; \mathbf{Z}_2) \to \mathbf{Z}_2[x]/(x^m)$, it follows that there exist polynomials $p(x), q(x), r(x) \in \mathbf{Z}_2[x]$, $p(0) = q(0) = r(0) = 1$, $\deg p(x) \leqslant k$, $\deg q(x)$, $\deg r(x) \leqslant n - k$, such that:

$$p(x) \cdot q(x) \equiv (1 + x)^n \pmod{x^m}, \tag{3}$$

$$p(x) \cdot r(x) \equiv 1 \pmod{x^m}. \tag{4}$$

Our theorems will follow by considering the above congruences.

Proof of Theorem 1 Suppose W is an m-dimensional k-space of real $n \times n$ matrices. If $m \geqslant n + 1$ then (3) and (4) imply that $p(x)q(x) = (1 + x)^n$ and $p(x)r(x) = 1$. Hence $p(x) = 1$ and so $q(x) = (1 + x)^n$, contradicting $\deg q(x) \leqslant n - k$. Therefore $\dim W = m \leqslant n$.

Proof of Theorem 2 Suppose now that $5 \leqslant \dim W = n$, n is a prime and 2 generates $\mathbf{Z}_n^*$. We shall need the following:

LEMMA (see [2, p. 75]) *If 2 generates $\mathbf{Z}_n^*$, then $1 + x + \cdots + x^{n-1}$ is irreducible in $\mathbf{Z}_2[x]$.* ■

We now consider (3) and (4) with $m = n$.

If $p(x)r(x) = 1$, then $p(x) = 1$ and so, by (3), $\deg q(x) \geqslant n - 1$. But $\deg q(x) \leqslant n - k$, hence $k = 1$. Otherwise $p(x)r(x) = 1 + x^n$, $\deg p(x) = k$, $\deg r(x) = n - k$, and hence, by the Lemma, either $p(x) = 1 + x$ and so $k = 1$, or $p(x) = 1 + x + \cdots + x^{n-1}$. Since $1 + x + \cdots + x^{n-1}$ is coprime to $(1 + x)^n$, (3) implies that $q(x) = 1$ and $p(x) =$

$(1+x)^n + x^n$. Therefore

$$\sum_{i=0}^{n-1} x^i = \sum_{i=0}^{n-1} \binom{n}{i} x^i,$$

and so

$$\binom{n}{i} \equiv 1 \pmod 2 \qquad \text{for} \quad 0 \leqslant i \leqslant n-1,$$

which, by Lucas' Theorem implies that $n = 2^l - 1$ for some l. But then the order of 2 in $\mathbf{Z}_n^*$ is $\leqslant l < n-1$ contradicting our assumptions. ■

Remarks

1. Theorems 1 and 2 are from the author's thesis [3].
2. Theorem 1 carries over to the complex case using Chern classes. Westwick [6], using the results of [5], has obtained rather tight bounds on the dimension of k-spaces of complex matrices.
3. L. Beasley has recently proved Theorem 1 for arbitrary fields.

References

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