# On Two Extremal Matrix Problems

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Dedicated to Alan J. Hoffman on the occasion of his 65th birthday.

Submitted by Alexander Schrijver

ABSTRACT

Let $W$ be a linear subspace of symmetric $n \times n$ matrices whose rank is at most $t$. It is shown that if the underlying field has more than $n$ elements, then:

$$\dim W \leqslant \begin{cases} \max\left\{\binom{t+1}{2}, kn - \binom{k}{2}\right\}, & t = 2k, \\ \max\left\{\binom{t+1}{2}, kn - \binom{k}{2} + 1\right\}, & t = 2k+1. \end{cases}$$

Let $L$ be an affine space of $n \times n$ matrices all having rank at least $k$. It is shown that if the underlying field is algebraically closed, then

$$\dim L \leqslant n^2 - \binom{k+1}{2}$$

The last result is applied to a problem of Valiant concerning permanents and determinants.

## 1. INTRODUCTION

Let $M_n(F)$ denote the space of $n \times n$ matrices over a field $F$, and let $H_n(F)$ be the subspace of symmetric $n \times n$ matrices.

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In Section 2 we give an upper bound for the dimension of linear subspaces of $H_n(F)$ having a bounded rank. Let $f_F(n,t)$ be the maximal dimension of a linear subspace $W \subseteq H_n(F)$ such that $\operatorname{rank}(A) \leqslant t$ for all $A \in W$.

EXAMPLE 1.

$$W_1(n,t) = \{ A \in H_n(F) : A(i,j) = 0 \text{ if } i > t \text{ or } j > t \}.$$

EXAMPLE 2.

$$W_2(n,2k) = \{ A \in H_n(F) : A(i,j) = 0 \text{ if } i > k \;\&\; j > k \},$$

$$W_2(n,2k+1) = \{ A \in H_n(F) : A(i,j) = 0 \text{ if } i > k \;\&\; j > k \;\&\; (i,j) \neq (k+1,k+1) \}.$$

Clearly $\operatorname{rank}(A) \leqslant t$ for any $A \in W_i(n,t)$.

THEOREM 1. *If* $|F| \geqslant \min\{t+3, n+1\}$, *then*

$$f_F(n,t) = \max\{\dim W_1(n,t), \dim W_2(n,t)\}$$

$$= \begin{cases} \max\left\{\binom{t+1}{2}, kn - \binom{k}{2}\right\}, & t = 2k, \\ \max\left\{\binom{t+1}{2}, kn - \binom{k}{2} + 1\right\}, & t = 2k+1. \end{cases}$$

In Section 3 we consider $h_F(n,k)$—the maximal dimension of an affine subspace of $M_n(F)$ which contains only matrices of rank at least $k$.

EXAMPLE.

$$L(n,k) = \{ A \in M_n(F) : A(i,j) = \delta_{ij} \text{ for } 1 \leqslant i \leqslant j \leqslant k \}.$$

THEOREM 2. *If $F$ is algebraically closed or $F = \mathbf{R}$, then*

$$h_F(n,k) = \dim L(n,k) = n^2 - \binom{k+1}{2}.$$

We conclude with an application of Theorem 2 to a problem of Valiant concerning representations of permanents in terms of determinants.

## 2. SPACES OF SYMMETRIC MATRICES OF BOUNDED RANK

We shall deduce Theorem 1 from a combinatorial lower bound on the maximal rank in a subspace of symmetric matrices (Theorem 3) and a simple extremal graph theoretic result (Theorem 4).

We shall use the following notation: $[n]^{\leqslant 2}$ will denote the collection of all nonempty subsets of $[n] = \{1,\dots,n\}$ of size $\leqslant 2$. $[n]^{(2)}$ will denote the complete graph on $[n]$. For a matrix $A \in H_n(F)$ we define $q(A) = \{i_0, j_0\} \in [n]^{\leqslant 2}$ where $(i_0, j_0) = \min\{(i,j): A(i,j) \neq 0\}$, and the minimum is taken with respect to the lexicographic ordering of $[n]\times[n]$ $[(i,j) < (i_1, j_1)$ iff $i < i_1$ or $i = i_1$ & $j < j_1]$.

A collection $\mathscr{B} = \{B_1,\dots,B_m\}$ is called a *matching* if the $B_i$'s are pairwise disjoint. For a graph with loops $\mathscr{B} \subseteq [n]^{\leqslant 2}$ we denote

$$\mu(\mathscr{B}) = \max\left\{\sum_{B\in\mathscr{B}'} |B| : \mathscr{B}' \subseteq \mathscr{B} \text{ is a matching}\right\}.$$

With a collection of matrices $\mathscr{A} = \{A_1,\dots,A_m\} \subset H_n(F)$ we associate a graph with loops $\mathscr{B} = \{q(A_1),\dots,q(A_m)\} \subseteq [n]^{\leqslant 2}$.

THEOREM 3. *If $|F| \geqslant \mu(\mathscr{B}) + 1$, then* $\mathrm{Span}(\mathscr{A})$ *contains a matrix of rank $\geqslant \mu(\mathscr{B})$.*

*Proof.* It suffices to show that if $\mu(\mathscr{B}) = n$ and $\{q(A_i)\}_{i=1}^m$ forms a partition of $[n]$, then $\mathrm{Span}(\mathscr{A})$ contains a nonsingular matrix. [The case $\mu(\mathscr{B}) < n$ reduces to the above, by considering the $\mu(\mathscr{B}) \times \mu(\mathscr{B})$ minors determined by a maximum matching in $\mathscr{B}$.]

Let $f(x_1,\dots,x_m) = \det(\sum_{i=1}^m x_i A_i)$. We shall show that the coefficient of $x_1^{a_1}\cdots x_m^{a_m}$ in $f$ is nonzero, where $a_i = |q(A_i)|$, $1 \leqslant i \leqslant m$.

Clearly this coefficient equals $\Sigma \det A$, where $A$ runs through all matrices of order $n$ which have exactly $a_i$ rows "belonging" to $A_i$, for all $1 \leqslant i \leqslant m$. More formally, consider two functions $f_1, f_2 : [m] \to [n]$ which satisfy

$$f_1(t) < f_2(t) \quad \text{if } a_t = 2, \qquad f_1(t) = f_2(t) \quad \text{if } a_t = 1 \qquad (*)$$

and

$$\bigcup_{t=1}^{m} \{ f_1(t), f_2(t) \} = [n].$$

The coefficient of $x_1^{a_1} \cdots x_m^{a_m}$ equals $\Sigma \det A(f_1, f_2)$, where the summation extends over all pairs $f_1, f_2$ which satisfy $(*)$, and $A(f_1, f_2) \in M_n(F)$ is defined as follows: For all $1 \leqslant t \leqslant m$ and $i = 1,2$, the $f_i(t)$th row of $A(f_1, f_2)$ is the $f_i(t)$th row of $A_t$.

We may assume now that $q(A_t) = \{ i_t, j_t \}$, where $i_t \leqslant j_t$ (equality iff $a_t = 1$), and $1 = i_1 < \cdots < i_m$. It is clear that if $f_1(t) = i_t$ and $f_2(t) = j_t$ for $1 \leqslant t \leqslant m$, then $A(f_1, f_2)$ is a row permutation of an upper triangular matrix with nonzeros on the diagonal, and hence $\det A(f_1, f_2) \neq 0$. Theorem 3 will therefore follow from:

CLAIM 1. *If* $\det A(f_1, f_2) \neq 0$, *then* $f_1(t) = i_t$ *and* $f_2(t) = j_t$ *for all* $1 \leqslant t \leqslant m$.

*Proof.* First we note the following:

(a) Columns $1, \ldots, i_s - 1$ in $A_s$ contain only zeros.

(b) If $q < j_s$, then the $q$th row of $A_s$ contains zeros in entries $1, \ldots, i_s$.

(c) Rows $1, \ldots, i_s - 1$ in $A_s$ are all zero; hence $\det A(f_1, f_2) \neq 0$ implies that $f_2(s) \geqslant f_1(s) \geqslant i_s$ for all $1 \leqslant s \leqslant m$.

We shall prove Claim 1 by verifying that the following statement holds: If $\det A(f_1, f_2) \neq 0$, then for all $1 \leqslant p \leqslant n$:

(1) If $p = i_t$ then $f_1(t) = i_t$ and $f_2(t) \geqslant j_t$.

(2) If $p = j_t > i_t$ then $f_2(t) = j_t$.

The proof proceeds by induction on $p$: Let $p = 1 = i_1$. Since $A_1$ is the only $A_i$ whose first row is nonzero, we must have $f_1(1) = 1$. Similarly $f_2(1) \geqslant j_1$; otherwise the first column of $A(f_1, f_2)$ would be zero.

Assume now that $1 < p \leqslant n$, and consider two cases:

(1) $p = i_t$: If $p = f_1(s)$ for some $s$, then by (c), $p = f_1(s) \geqslant i_s$. If $p > i_s$, then by induction $f_1(s) = i_s < p$. Hence $i_s = p$ and so $s = t$ and $p = f_1(t)$.

Assume then that $p = f_2(s) > f_1(s)$. As before, it follows that $p > i_s$ and hence $p = f_2(s) \geqslant j_s$. The possibility $p > j_s$ is excluded by induction; therefore $i_t = p = j_s$ and so $i_s = j_s$, contradicting our assumption $f_1(s) < f_2(s)$. It remains to show that $f_2(t) \geqslant j_t$. Suppose, to the contrary, that $f_2(t) < j_t$, and let $B$ be the $n \times p$ matrix consisting of the first $p$ columns of $A(f_1, f_2)$.

LEMMA. *If the* $q$th *row of* $B$ *is nonzero, then*

$$q \in \{ f_2(s) : i_s < p \} \cup \{ f_1(s) : j_s < p \}.$$

*Proof.* Assume that $q = f_2(s)$. By (a), $i_s \leqslant p$. If $i_s = p = i_t$, then $s = t$, and so $q = f_2(s) = f_2(t) < j_t = j_s$. But then, by (b), the $q$th row of $A_s$ contains zeros in entries $1, \dots, i_s = p$, a contradiction. Therefore $i_s < p$.

Assume now that $q = f_1(s) < f_2(s)$. As before, $i_s \leqslant p$, and so by induction (and the beginning of the proof), $q = f_1(s) = i_s$. But the $i_s$th row of $A_s$ contains zeros in places $1, \dots, j_s - 1$; therefore it follows that $i_t = p \geqslant j_s$, and so $j_s < p$. ∎

The lemma implies that the number of nonzero rows in $B$ is at most $|\{ f_2(s) : i_s < p\} \cup \{ f_1(s) : j_s < p\}| = p - 1$. Hence $\operatorname{rank}(B) \leqslant p - 1$ and $A(f_1, f_2)$ must be singular, contradicting our assumption. Therefore $f_2(t) \geqslant j_t$, which completes the proof of case (1).

(2) $p = j_t > i_t$. We have to show that $f_2(t) = j_t$. If $p = f_1(s)$ for some $s$, then $i_s \leqslant p = j_t$; hence $i_s < p$ and so, by induction, $f_1(s) = i_s < p$, a contradiction. Therefore $p = f_2(s) > f_1(s)$. Again $i_s < p$, and by induction $p = f_2(s) \geqslant j_s$. Now $j_s < p$ is excluded by induction, so we conclude that $j_s = p = j_t$, $s = t$, and $f_2(t) = j_t$. ∎

Claim 1 shows that $f(x_1, \dots, x_m)$ is not identically zero, which together with the assumption on the size of $|F|$ proves Theorem 3. ∎

In the following we shall need a certain extremal property of $\mu$. For $1 \leqslant t \leqslant n$ let

$$u(n, t) = \max\left\{ |\mathscr{B}| : \mathscr{B} \subseteq [n]^{\leqslant 2} \text{ and } \mu(\mathscr{B}) \leqslant t \right\}.$$

EXAMPLE 1.

$$\mathscr{B}_1(n, t) = [t]^{\leqslant 2}.$$

EXAMPLE 2.

$$\mathscr{B}_2(n,2k) = [k]^{\leqslant 2} \cup \left\{ e \in [n]^{(2)} : |e \cap [k]| = 1 \right\},$$

$$\mathscr{B}_2(n,2k+1) = \mathscr{B}_2(n,2k) \cup \{ \{ k+1 \} \}.$$

Clearly $\mu(\mathscr{B}_i(n,t)) = t$.

The following theorem and its proof are similar to the analogous graphical case (see Problem 3.31 in [1]).

THEOREM 4.

$$u(n,t) = \max\left\{ |\mathscr{B}_1(n,t)|, |\mathscr{B}_2(n,t)| \right\}$$

$$= \begin{cases} \max\left\{ \binom{t+1}{2}, kn - \binom{k}{2} \right\}, & t = 2k, \\ \max\left\{ \binom{t+1}{2}, kn - \binom{k}{2} + 1 \right\}, & t = 2k+1. \end{cases}$$

*Proof.* Let $\mathscr{B} \subseteq [n]^{\leqslant 2}$ be a maximal collection satisfying $\mu(\mathscr{B}) \leqslant t$. Maximality implies that $\mu(\mathscr{B}) = t$ and that if $\{x\}, \{y\} \in \mathscr{B}$ then $\{x, y\} \in \mathscr{B}$. Let $G = \mathscr{B} \cap [n]^{(2)}$ be the graph without loops induced by $\mathscr{B}$; then clearly $\nu(G)$, the maximum number of disjoint edges in $G$, equals $\lfloor t/2 \rfloor$. We consider two cases:

(a) $t = 2k + 1$: Clearly $G$ is a maximal graph on $[n]$ satisfying $\nu(G) = k$. By Tutte's theorem (Exercise 7.27 in [5]) there exists a subset of vertices $S$, $|S| = s$, such that $c_1(G - S)$, the number of odd components of $G - S$, equals $n + s - 2k$. Denote by $A_1, \dots, A_{n+s-2k}$ ($B_1, \dots, B_l$) the odd (even) components of $G - S$. The maximality of $G$ implies that $G[S]$, $G[A_i]$, $G[B_j]$ are complete graphs (for all $i$ and $j$), and that all edges $e \in [n]^{(2)}$ which satisfy $|e \cap S| = 1$ necessarily belong to $G$. This structure of $G$ implies that at most one of the odd components of $G - S$, say $A_1$, contains vertices which are loops in $\mathscr{B}$. Denoting $|A_i| = a_i$, $|B_j| = b_j$, and using

$$\sum_{i=1}^{r} \binom{x_i}{2} \leqslant \binom{\sum_{i=1}^{r} x_i - r + 1}{2},$$

whenever $x_i \geqslant 1$, we obtain

$$|\mathscr{B}| \leqslant \binom{s+1}{2} + s(n-s) + \sum_{j=1}^{l} \binom{b_j+1}{2} + \binom{a_1+1}{2} + \sum_{i=2}^{n+s-2k} \binom{a_i}{2}$$

$$\leqslant \binom{s+1}{2} + s(n-s) + \binom{\sum_{j=1}^{l} b_j + \sum_{i=1}^{n+s-2k} a_i + 2k + 2 - n - s}{2}$$

$$= \binom{s+1}{2} + s(n-s) + \binom{2(k-s)+2}{2}.$$

This implies, since $0 \leqslant s \leqslant k$, that

$$|\mathscr{B}| \leqslant \max\left\{\binom{2k+2}{2}, kn - \binom{k}{2} + 1\right\}.$$

(b) $t = 2k$: As before, $\nu(G) = k$. By the Gallai-Edmonds theorem (Exercise 7.32 in [5]), there exists $S \subseteq [n]$, $|S| = s$, such that $2k = n + s - c_1(G-S)$ and $\nu(G-x) = \nu(G)$ for any vertex $x$ belonging to an odd component of $G - S$. Since $\mu(\mathscr{B}) = 2k$, this last property implies that none of the odd components of $G - S$ contains a singleton of $\mathscr{B}$. Denoting the components of $G - S$ as before, we obtain

$$|\mathscr{B}| \leqslant \binom{s+1}{2} + s(n-s) + \sum_{j=1}^{l} \binom{b_j+1}{2} + \sum_{i=1}^{n+s-2k} \binom{a_i}{2}$$

$$\leqslant \binom{s+1}{2} + s(n-s) + \binom{2(k-s)+1}{2},$$

and so

$$|\mathscr{B}| \leqslant max\left\{\binom{2k+1}{2}, kn - \binom{k}{2}\right\}.$$

Theorem 1 now follows from Theorems 3 and 4: Suppose $W = \mathrm{Span}\{A_1, \dots, A_m\} = \mathrm{Span}(\mathscr{A})$ is an $m$-dimensional subspace of $H_n(F)$ such that $\mathrm{rank}(A) \leqslant t$ for all $A \in W$. Performing a Gaussian elimination on $A_1, \dots, A_m$ (considered as $n^2$-dimensional vectors), we may assume that $\mathscr{B} = \{q(A_1), \dots, q(A_m)\}$ contains $m$ distinct edges (including loops). If $m > u(n,t)$, then $\mu(\mathscr{B}) \geqslant t+1$, and so we may choose $\mathscr{B}' \subseteq \mathscr{B}$ such that $t+1 \leqslant \mu(\mathscr{B}') \leqslant t+2$. Let $\mathscr{A}' \subseteq \mathscr{A}$ be the subcollection of matrices which corresponds to $\mathscr{B}'$, then by Theorem 3, $\mathrm{Span}(\mathscr{A}')$ contains a matrix of rank $\geqslant \mu(\mathscr{B}') \geqslant t+1$, contradicting our assumptions. Therefore $\dim W = m \leqslant u(n,t)$, and so Theorem 1 follows from Theorem 4.

Remarks.

(1) Denote by $g_F(n,2k)$ the maximal dimension of a subspace of skew-symmetric $F$-matrices of rank $\leqslant 2k$. Theorem 3 and the graphical analogue of Theorem 4 imply that if $|F| \geqslant \min\{2k+3, n+1\}$ and $\mathrm{char}\, F \neq 2$ then

$$g_F(n,2k) = \max\left\{\binom{2k}{2}, kn - \binom{k+1}{2}\right\}.$$

(2) The maximal dimension of a subspace of matrices of bounded rank is discussed by Flanders [2] (see also [6]).

## 3. THE MINIMAL RANK IN AN AFFINE SUBSPACE OF MATRICES

We assume throughout this section that either $F = \mathbf{R}$ or $F$ is algebraically closed.

In the proof of Theorem 2 we shall make use of the following:

Claim 2. *If a linear subspace $W \subseteq M_r(F)$ satisfies*

$$\dim W > \binom{r}{2},$$

*then $W$ contains a matrix with a nonzero eigenvalue which belongs to the field.*

*Proof.*

(a) $F = \mathbf{R}$: Clearly

$$\dim[W \cap H_r(\mathbf{R})] \geqslant \dim W + \dim H_r(\mathbf{R}) - \dim M_r(\mathbf{R})$$
$$> \binom{r}{2} + \binom{r+1}{2} - r^2 = 0.$$

Hence $W$ contains a nonzero symmetric matrix which, of course, has a nonzero real eigenvalue.

(b) $F$ *is algebraically closed*: If $W$ does not contain a matrix with a nonzero eigenvalue, then all matrices in $W$ are nilpotent, and hence by a theorem of Gerstenhaber [4],

$$\dim W \leqslant \binom{r}{2}. \qquad \blacksquare$$

We can now show that if $L$ is an $n^2 - \binom{k+1}{2} + 1$ dimensional affine subspace of $M_n(F)$, then $L$ contains a matrix of rank at most $k-1$. By (downward) induction on $k$, $L$ contains a matrix $A_0$ of rank $\leqslant k$. If $\operatorname{rank}(A_0) \leqslant k-1$ then we are done, so we assume $\operatorname{rank}(A_0) = k$. Changing bases, we may take $A_0$ to be

$$\begin{pmatrix} I_k & 0 \\ 0 & 0 \end{pmatrix},$$

where $I_k$ is the identity matrix of order $k$.

Let $W$ be the linear subspace of $M_n(F)$ for which $L = A_0 + W$, and let

$$U = \left\{ B \in M_k(F) : \begin{pmatrix} B & 0 \\ 0 & 0 \end{pmatrix} \in W \right\}.$$

Clearly

$$\dim U \geqslant \dim W + \dim M_k(F) - \dim M_n(F)$$
$$\geqslant \left(n^2 - \binom{k+1}{2} + 1\right) + k^2 - n^2 = \binom{k}{2} + 1,$$

and so, by Claim 2, there exists a matrix $B \in U$ with an eigenvalue $0 \neq \lambda \in F$, i.e. $\det(I_k - \lambda^{-1}B) = 0$. Now

$$B_0 = \begin{pmatrix} B & 0 \\ 0 & 0 \end{pmatrix} \in W,$$

and so $A_0 - \lambda^{-1}B_0$ belongs to $L$ and satisfies

$$\operatorname{rank}(A_0 - \lambda^{-1}B_0) = \operatorname{rank}\begin{pmatrix} I_k - \lambda^{-1}B & 0 \\ 0 & 0 \end{pmatrix} \leqslant k-1,$$

which proves Theorem 2.

We conclude with an application of Theorem 2: Let $F$ be an infinite field of characteristic $\neq 2$. The permanent of order $n$ is said to be a *projection* of the determinant of order $m$ if there exists an $m \times m$ matrix $f$, whose entries are either elements of $F$ or variables $\pm x_{ij}$, $1 \leqslant i, j \leqslant n$, such that $\operatorname{per} x_{ij} = \det f$ holds as a polynomial identity in $F[x_{11}, \ldots, x_{nn}]$.

Let $p(n)$ denote the least $m$ for which such a matrix exists. Valiant [7] has shown that $p(n) = O(n^2 2^n)$ and conjectured that $p(n)$ grows superpolynomially in $n$. von zur Gathen [3] has recently proved that $p(n) \geqslant \sqrt{\frac{8}{7}}\, n - 1$. In Theorem 5 we slightly improve this bound.

We shall apply yet another theorem of von zur Gathen:

THEOREM [3]. *If $F$ is an infinite field of characteristic $\neq 2$ and $f\colon M_n(F) \to M_m(F)$ is a polynomial mapping which satisfies* $\operatorname{per} X = \det f(X)$, *then* $\operatorname{rank} f(A) \geqslant m - 1$ *for all* $A \in M_n(F)$.

We shall also need the following easy fact (which may be proved by induction on the maximal order of a nonzero subpermanent of $A$):

CLAIM 3. *If $0 \neq A \in M_n(F)$, then there exists $B \in M_n(F)$ for which* $\operatorname{per}(A + B) \neq \operatorname{per} B$.

THEOREM 4. *Let $F$ be an infinite field of characteristic $\neq 2$, and let $f\colon M_n(F) \to M_m(F)$ be an affine map which satisfies* $\operatorname{per} X = \det f(X)$. *Then $m \geqslant \sqrt{2}\, n - 1$.*

*Proof.* We may assume that $F$ is algebraically closed. Next we note that $f$ is injective: If $A \neq 0$ and $f(A) = f(0)$, then by Claim 3 there exists $B$ satisfying $\operatorname{per} B \neq \operatorname{per}(A + B)$. On the other hand $\operatorname{per} B = \det f(B) =$

det $f(A+B) = \text{per}(A+B)$, a contradiction. It follows that $L = \{f(X): X \in M_n(F)\}$ is an $n^2$-dimensional affine subspace of $M_m(F)$, and according to von zur Gathen's theorem all ranks in $L$ are at least $m-1$.

Theorem 2 now implies that

$$n^2 = \dim L \leqslant m^2 - \binom{m}{2} = \binom{m+1}{2},$$

and so $m \geqslant \sqrt{2}\, n - 1$. ■

REMARK. A result similar to Theorem 4 has also been obtained by L. Babai and A. Seress (see [3]).

## REFERENCES

1 B. Bollobas, *Graph Theory — An Introductory Course*, Springer-Verlag, 1978.
2 H. Flanders, On subspaces of linear transformation of bounded rank, *J. London Math. Soc.* 37:10–16 (1962).
3 J. von zur Gathen, Permanent and determinant, *Linear Algebra Appl.* 96:87–100 (1987).
4 M. Gerstenhaber, On nilalgebras and linear varieties of nilpotent matrices, *Amer. J. Math.* 80:614–622 (1958).
5 L. Lovasz, *Combinatorial Problems and Exercises*, North Holland, 1979.
6 R. Meshulam, On the maximal rank in a subspace of matrices, *Quart. J. Math.* 36:225–229 (1985).
7 L. Valiant, The complexity of computing the permanent, *Theoret. Comput. Sci.* 8:189–201 (1979).

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