Spaces of p-Vectors of Bounded Rank

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Abstract

Let V be an n-dimensional space over an infinite field of characteristic different from 2. The *rank* of $w \in \bigwedge^p V$ is the minimal dimension of a subspace $U \subset V$ such that $w \in \bigwedge^p U$. Extending a well-known result on linear spaces in the Grassmannian, it is shown that if $p \leq k < n$ then the maximal dimension of a subspace $W \subset \bigwedge^p V$ such that rank $w \leq k$ for all $w \in W$ is

$$\max\{\binom{k+\epsilon}{p}, \binom{m}{p} + (k-m)(n-m)\}$$

where $\epsilon = 1$ if $k = p$ or $p = 2|k$, $\epsilon = 0$ otherwise, and m satisfies $\binom{m-1}{p-1} + m \leq k \leq \binom{m}{p-1} + m$.

1 Introduction

Let V be an n-dimensional vector space over an infinite field $\mathbb{F}$ of characteristic different from 2, and let $\bigwedge^p V$ denote the p-th exterior power of V. The *enveloping space* $E(w)$ of a p-vector $w \in \bigwedge^p V$ is the unique minimal subspace $U \subset V$ such that $w \in \bigwedge^p U$. The *rank* of w is defined by rank $w = \dim E(w)$.

In this paper we are concerned with the maximal dimension of linear subspaces of $\bigwedge^p V$ consisting of p-vectors of bounded rank. Our work is motivated by two results on linear spaces in certain algebraic varieties:

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a) Let $G_p(V)$ denote the Grassmannian of all $p-$dimensional linear subspaces of V. We identify $G_p(V)$ with its image under the Plücker embedding in the projective space $\mathbb{P}(\bigwedge^p V)$. The maximal linear spaces in $G_p(V)$ are classically described (see e.g. Exercise 6.9 in [8]) by the following

Proposition 1.1 *If M is a maximal linear space in $G_p(V) \subset \mathbb{P}(\bigwedge^p V)$ then either $M = \{\Lambda \in G_p(V) \ : \ \Lambda \subset U\}$ for some $(p+1)$-dimensional subspace $U \subset V$, or $M = \{\Lambda \in G_p(V) \ : \ \Lambda \supset U'\}$ for some $(p-1)$-dimensional $U' \subset V$.*

While easy to prove, Proposition 1.1 plays a role in a variety of applications (see e.g. Theorem 10.19 in [8], Theorem 2.4 in [5] and [13]).
A p-vector $w \in \bigwedge^p V$ is *decomposable* if rank $w = p$. Proposition 1.1 implies that if $W \subset \bigwedge^p V$ is a linear space of decomposable p-vectors then either $W \subset \bigwedge^p U$ for some (p+1)-dimensional $U \subset V$ or $W \subset \{v_1 \wedge \cdots \wedge v_{p-1} \wedge v \ : \ v \in V\}$ for some $v_1, \ldots, v_{p-1} \in V$. In particular

$$\dim W \leq \max\{p+1, n-p+1\} \ . \tag{1}$$

b) Let $M_n(\mathbb{F})$ denote the space of $n \times n$ matrices over $\mathbb{F}$. Linear spaces of matrices of bounded rank have been studied extensively both for their own interest and in connection with certain preserver problems (see e.g. [1, 2, 5, 3]). Flanders [6] (see also [11]) proved that if $W \subset M_n(\mathbb{F})$ is a linear subspace such that rank $A \leq k$ for all $A \in W$, then $\dim W \leq kn$, and equality holds only for $W = E \otimes \mathbb{F}^n$ or $W = \mathbb{F}^n \otimes E$ where E is a k-dimensional subspace of $\mathbb{F}^n$. A skew-symmetric counterpart of Flanders' result is given in [12]: If $W \subset M_n(\mathbb{F})$ is a linear subspace of skew-symmetric matrices of rank at most $k = 2t < n$ then

$$\dim W \leq \max\{\binom{2t+1}{2}, \binom{t}{2} + t(n-t)\} \ . \tag{2}$$

Our main result is a common extension of (1) and (2) to subspaces of $p-$vectors of bounded rank.
For a subspace $W \subset \bigwedge^p V$ let $\rho(W) = \max\{\text{rank } w \ : \ w \in W\}$, and for $p \leq k < n$ let $f_p(n,k) = \max\{\dim W \ : \ W \subset \bigwedge^p V \ , \ \rho(W) \leq k\}$.
$f_p(n,p)$ and $f_2(n,k)$ are given by (1) and (2) respectively. Here we determine $f_p(n,k)$ in general.

For $p \le k \le n$ let $\epsilon_p(k) = 1$ if $p = k$ or $p = 2|k$, and $\epsilon_p(k) = 0$ otherwise. Let $m_p(k)$ denote the unique m such that $\binom{m-1}{p-1} + m \le k \le \binom{m}{p-1} + m$.
Define $a_p(k) = \binom{k+\epsilon_p(k)}{p}$, $b_p(n,k) = \binom{m_p(k)}{p} + (k - m_p(k))(n - m_p(k))$ and $h_p(n,k) = \max\{a_p(k), b_p(n,k)\}$.

Theorem 1.2 *$f_p(n,k) = h_p(n,k)$ for $p \le k < n$.*

We recall some definitions from exterior algebra (for more details see [10]). We assume throughout that $V = \mathbb{F}^n$ and denote the standard basis by $e_1, \ldots, e_n$. For $x = (x_1, \ldots, x_n), y = (y_1, \ldots, y_n) \in V$ let $(x,y) = \sum_{i=1}^n x_i y_i$. The standard non-degenerate pairing on $\bigwedge^p V$ is given by $(u_1 \wedge \cdots \wedge u_p, v_1 \wedge \cdots \wedge v_p) = \det[(u_i, v_j)]_{i,j=1}^p$. A p-vector $w \in \bigwedge^p V$ defines a unique linear map $\llcorner w \ : \ \bigwedge^q V \to \bigwedge^{q-p} V$ which satisfies $(u \llcorner w, v) = (u, w \wedge v)$ for all $u \in \bigwedge^q V$, $v \in \bigwedge^{q-p} V$. The enveloping space of $w \in \bigwedge^p V$ is given explicitly by $E(w) = \{w \llcorner u \ : \ u \in \bigwedge^{p-1} V\}$.
The rank of a non-decomposable $w \in \bigwedge^p V$ is at least $p+2$, and the rank of $w \in \bigwedge^2 V$ is always even. It follows that rank $w \ne k + \epsilon_p(k)$ for all k.
Denote $[n] = \{1, \ldots, n\}$ and $[a,b] = \{a, a+1, \ldots, b\}$. For a set X let $\binom{X}{p} = \{A \subset X : |A| = p\}$. For $F = \{i_1 < \cdots < i_p\} \subset [n]$ let $e_F = e_{i_1} \wedge \cdots \wedge e_{i_p}$. If $|F| = p \le q = |F'|$ then $e_{F'} \llcorner e_F = \pm e_{F'-F}$ if $F \subset F'$, and zero otherwise. With a p-graph $\mathcal{F} \subset \binom{[n]}{p}$ we associate the subspace $W(\mathcal{F}) = \mathrm{Span}\{e_F \ : \ F \in \mathcal{F}\} \subset \bigwedge^p V$.

Let $p \le k < n$ and let $\epsilon = \epsilon_p(k)$, $m = m_p(k)$. The lower bound on $f_p(n,k)$ is provided by the following examples of spaces of rank at most k:
1. Let $\mathcal{A}_p(k) = \binom{[k+\epsilon]}{p}$, then $\dim W(\mathcal{A}_p(k)) = |\mathcal{A}_p(k)| = a_p(k)$. If $w \in W(\mathcal{A}_p(k))$ then $E(w) \subset \mathrm{Span}\{e_i \ : \ 1 \le i \le k+\epsilon\}$, hence rank $w \le k$.
2. Choose an arbitrary subset $\{G_1, \ldots, G_{k-m}\} \subset \binom{[m]}{p-1}$, and let

$$\mathcal{B}_p(n,k) = \binom{[m]}{p} \cup \{F \in \binom{[n]}{p} \ : \ F \cap [m] \in \{G_i\}_{i=1}^{k-m} \ \} .$$

$\dim W(\mathcal{B}_p(n,k)) = |\mathcal{B}_p(n,k)| = b_p(n,k)$. If $w \in W(\mathcal{B}_p(n,k))$ then there exist $v_1, \ldots, v_{k-m} \in V$ such that

$$w = \sum_{F \in \binom{[m]}{p}} c_F \ e_F + \sum_{i=1}^{k-m} c_i' \ e_{G_i} \wedge v_i \quad .$$

Hence $E(w) \subset \mathrm{Span}\{e_1, \ldots, e_m, v_1, \ldots, v_{k-m}\}$ and rank $w \le k$.
It follows that $f_p(n,k) \ge h_p(n,k)$.

The proof of the upper bound in Theorem 1.2 involves some multilinear algebra and extremal finite set theory. In section 2 we use the notion of weak matching in a uniform hypergraph to give a combinatorial lower bound on the maximal rank in a subspace of p-vectors. In section 3 we prove an extremal result for weak matchings which then yields the upper bound on $f_p(n,k)$. We conclude with a problem concerning the uniqueness of the extremal examples.

2 A lower bound on the maximal rank

A p-graph $\mathcal{H} \subset \binom{[n]}{p}$ is a *weak matching* (abbreviated w-matching) if $|H_1 \cap H_2| < p-1$ for all $H_1 \neq H_2 \in \mathcal{H}$. A w-matching $\mathcal{H}$ *covers* a set X if $\cup_{H \in \mathcal{H}} H \supset X$. For $\mathcal{F} \subset \binom{[n]}{p}$ let

$$\lambda(\mathcal{F}) = \max\{| \cup_{H \in \mathcal{H}} H| \ : \ \mathcal{H} \subset \mathcal{F} \ \text{ is w-matching }\}.$$

The colexicographic order on $\binom{[n]}{p}$ is given by $A \prec B$ if $\max(A \triangle B) \in B$. For $A \subset [n]$ let $\alpha(A) = \sum_{i \in A} 2^i$, then $A \prec B$ if and only if $\alpha(A) < \alpha(B)$. For $0 \neq w = \sum_{|F|=p} c_F e_F \in \bigwedge^p V$ let $q(w) = \max\{F \ : \ c_F \neq 0\}$ denote the index of the leading non-zero coefficient of w in the colexicographic order. With a subspace $W \subset \bigwedge^p V$ we associate the p-graph $\mathcal{F}_W = \{q(w) : 0 \neq w \in W\}$. Clearly $|\mathcal{F}_W| = \dim W$. A link between the maximal rank in W and weak matchings in $\mathcal{F}_W$ is given by the following

Theorem 2.1 $\rho(W) \geq \lambda(\mathcal{F}_W)$.

Proof: Let $\mathcal{F} = \{F_1, \ldots, F_t\} \subset \mathcal{F}_W$ be a w-matching such that $A = \cup_{i=1}^t F_i$ has cardinality $\lambda(\mathcal{F}_W)$. We may assume that the F_i's are distinct and $F_i = q(w_i)$, where $w_i = \sum_{H \preceq F_i} c_i(H) e_H \in W$ and $c_i(F_i) = 1$.
Let $[t]^A$ denote the set of mappings from A to $[t]$. Fix a $\phi \in [t]^A$ such that $j \in F_{\phi(j)}$ for all $j \in A$, and let $G_j = F_{\phi(j)} - \{j\}$.
Let $x_1, \ldots, x_t$ be variables and let $w(x) = \sum_{i=1}^t x_i w_i$. We'll show that there exists an $a = (a_1, \ldots, a_t) \in \mathbb{F}^t$ such that rank $w(a) \geq |A| = \lambda(\mathcal{F}_W)$.
Let $B = [n] - A$ and define $u(x) = e_B \wedge \bigwedge_{j \in A} (w(x) \llcorner e_{G_j})$. Then $u(x) = p(x) e_{[n]}$ where $p(x) \in \mathbb{F}[x_1, \ldots, x_t]$. It suffices to show that $p(x) \neq 0$. Indeed, this will imply that there exists an $a \in \mathbb{F}^t$ such that $u(a) \neq 0$ and in particular $\bigwedge_{j \in A} (w(a) \llcorner e_{G_j}) \neq 0$. Since the lefthand side belong to $\bigwedge^{|A|} E(w(a))$, it follows that rank $w(a) = \dim E(w(a)) \geq |A| = \lambda(\mathcal{F})$.

We'll show that the monomial $\prod_{j\in A} x_{\phi(j)} = \prod_{i=1}^{t} x_i^{|\phi^{-1}(i)|}$ appears in $p(x)$ with a non-zero coefficient. Now

$$u(x) = e_B \wedge \bigwedge_{j\in A} (\sum_{i=1}^{t} x_i w_i \llcorner e_{G_j}) = \sum_{\psi\in[t]^A} (e_B \wedge \bigwedge_{j\in A} (w_{\psi(j)} \llcorner e_{G_j}) \prod_{j\in A} x_{\psi(j)} .$$

Let $M = \{\psi \in [t]^A \ : \ \prod_{j\in A} x_{\psi(j)} = \prod_{j\in A} x_{\phi(j)}\}$, then for $\psi \in M$

$$e_B \wedge \bigwedge_{j\in A} (w_{\psi(j)} \llcorner e_{G_j}) = e_B \wedge \bigwedge_{j\in A} (\sum_{H_j \preceq F_{\psi(j)}} c_{\psi(j)}(H_j)(e_{H_j} \llcorner e_{G_j}) =$$

$$\sum_{\{(H_j)_{j\in A} \ : \ H_j \preceq F_{\psi(j)}\}} \prod_{j\in A} c_{\psi(j)}(H_j) \ e_B \wedge \bigwedge_{j\in A} (e_{H_j} \llcorner e_{G_j}) \ .$$

If $e_B \wedge \bigwedge_{j\in A}(e_{H_j} \llcorner e_{G_j}) \neq 0$, then $H_j \supset G_j$ for all $j \in A$, and $\bigcup_{j\in A}(H_j - G_j) = A$. Combining this with $\sum_{j\in A} \alpha(F_{\psi(j)}) = \sum_{j\in A} \alpha(F_{\phi(j)})$ we obtain

$$\sum_{j\in A} 2^j = \sum_{j\in A} (\alpha(H_j) - \alpha(G_j)) \leq \sum_{j\in A} (\alpha(F_{\psi(j)}) - \alpha(G_j)) =$$

$$\sum_{j\in A} (\alpha(F_{\phi(j)}) - \alpha(G_j)) = \sum_{j\in A} 2^j \ .$$

Hence $H_j = F_{\psi(j)}$ for all $j \in A$. It follows that the coefficient of $\prod_{j\in A} x_{\phi(j)}$ in $u(x)$ is

$$\sum_{\psi\in M} e_B \wedge \bigwedge_{j\in A} (e_{F_{\psi(j)}} \llcorner e_{G_j}). \tag{3}$$

Let $j \in A$. Since $\mathcal{F}$ is a w-matching G_j is contained in a unique F_i, namely $F_{\phi(j)}$. It follows that the unique nonzero summand in (3) is the one corresponding to $\psi = \phi$, hence the coefficient of $\prod_{j\in A} x_{\phi(j)}$ in $u(x)$ is

$$e_B \wedge \bigwedge_{j\in A} (e_{F_{\phi(j)}} \llcorner e_{G_j}) = \pm e_{[n]} \ \neq 0 \ .$$

□

We can now give a combinatorial upper bound on $f_p(n,k)$.
Let $g_p(n,k) = \max\{|\mathcal{F}| \ : \ \mathcal{F} \subset \binom{[n]}{p} \ , \ \lambda(\mathcal{F}) \leq k\}$.

Corollary 2.2 $f_p(n,k) \leq g_p(n,k)$.

Proof: Let W be subspace of $\bigwedge^p V$ such that $\rho(W) \leq k$ and $\dim W = f_p(n,k)$. By Theorem 2.1 $\lambda(\mathcal{F}_W) \leq \rho(W) \leq k$, hence $f_p(n,k) = \dim W = |\mathcal{F}_W| \leq g_p(n,k)$.

□

3 An extremal problem for weak matchings

To complete the proof of Theorem 1.2 it suffices, by Corollary 2.2, to show the following combinatorial result.

Theorem 3.1 *$g_p(n,k) = h_p(n,k)$ for $p \le k < n$.*

The graphical case $p = 2$ of Theorem 3.1 is due to Erdős and Gallai [4]. The proof in [4] applies Tutte's Theorem on graph matchings (See [9]). Unfortunately we do not have a Tutte's type characterization of weak matchings in hypergraphs, so for the general case we must proceed differently.
For the lower bound note that $\lambda(\mathcal{A}_p(k)), \lambda(\mathcal{B}_p(n,k)) \le k$ hence $g_p(n,k) \ge h_p(n,k)$. The first step in the proof of the upper bound is an application of the shifting method (see e.g. [7]).
Let $1 \le i < j \le n$. The (i,j)-shift of a hypergraph $\mathcal{F} \subset 2^{[n]}$ is given as follows: For $F \in \mathcal{F}$ let

$$S_{i,j}(F) = \begin{cases} F - \{j\} \cup \{i\} & \text{if } \ j \in F \ , \ i \notin F \ , \ F - \{j\} \cup \{i\} \notin \mathcal{F} \\ F & \text{otherwise} \end{cases}$$

and let $S_{i,j}(\mathcal{F}) = \{S_{i,j}(F) \ : \ F \in \mathcal{F}\}$. Note that $|S_{i,j}(\mathcal{F})| = |\mathcal{F}|$.

Claim 3.2 *For $1 \le i < j \le n$, $\lambda(S_{i,j}(\mathcal{F})) \le \lambda(\mathcal{F})$.*

Proof: We may assume $(i,j) = (1,2)$ and abbreviate $S_{1,2} = S$.
Let $\mathcal{G} = \{G_1, \ldots, G_t\} \subset S(\mathcal{F})$ be a w-matching such that $A = \bigcup_{i=1}^t G_i$ satisfies $|A| = \lambda(S(\mathcal{F}))$, and let $G_i = S(F_i)$. If $F_i = G_i$ for all $1 \le i \le t$ then $\mathcal{G} \subset \mathcal{F}$ and we are done. Suppose then that $F_{j_0} \ne G_{j_0}$, and let
$I = \{1 \le i \le t \ : \ 1 \notin G_i \ , \ 2 \in G_i\}$, $J = [t] - I$.
If $i \in I$ then $F_i = G_i$ and $F_i' = F_i - \{2\} \cup \{1\} \in \mathcal{F}$. Let
$I_1 = \{i \in I : |F_i' \cap F_j| < p - 1 \text{ for all } j \in J\}$,
$I_2 = \{i \in I - I_1 : |F_i \cap F_j| < p - 1 \text{ for all } j \in J\}$. Let $\mathcal{H} = \{F_i\}_{i \in J \cup I_2} \cup \{F_i'\}_{i \in I_1} \subset \mathcal{F}$. By the construction $\mathcal{H}$ is a w-matching. Let $B = \bigcup_{H \in \mathcal{H}} H$. The following observations imply that $\lambda(\mathcal{F}) \ge |B| \ge |A| = \lambda(S(\mathcal{F}))$:
1. $|B \cap \{1,2\}| \ge |A \cap \{1,2\}|$: Note that $2 \in F_{j_0} \subset B$. If $2 \notin A$ we are done, else $2 \in G_l$ for some $1 \le l \le t$. If $1 \in G_l$ then $l \in J$ and $\{1,2\} \subset G_l = F_l \subset B$. Otherwise $l \in I$. If $l \in I_1$ then $1 \in F_l' \subset B$. Otherwise $|F_l' \cap F_j| = p - 1$ for some $j \in J$ and $1 \in F_j \subset B$. Thus $2 \in A$ implies $B \supset \{1,2\}$.
2. $B \cap \{3, \ldots, n\} = A \cap \{3, \ldots, n\}$: It suffices to show that $G_i \cap \{3, \ldots, n\} \subset B$ for all $i \in I - (I_1 \cup I_2)$. Let $j_1 \ne j_2 \in J$ such that $C_1 = F_i' \cap F_{j_1}$, $C_2 =$

$F_i \cap F_{j_2}$ satisfy $|C_1| = |C_2| = p-1$. If $C_1 - \{1\} = C_2 - \{2\} = C$, then $|G_{j_1} \cap G_{j_2}| = |\{1\} \cup C| = p-1$, a contradiction. Therefore $C_1 - \{1\} \neq C_2 - \{2\}$, hence $G_i \cap \{3,\ldots,n\} \subset F_i \subset F_{j_1} \cup F_{j_2} \subset B$.

$\square$

We proceed with the proof of Theorem 3.1: Let $\mathcal{F} \subset \binom{[n]}{p}$ be a p-graph such that $\lambda(\mathcal{F}) \leq k < n$ and $|\mathcal{F}| = g_p(n,k)$. Since $h_p(n,p) = h_p(n,p+1)$ and $h_2(n,k) = h_2(n,k+1)$ when k is even, we may assume that $k \geq p+1$ and that k is odd if $p=2$. By finitely many applications of Claim 3.2 we may assume that $\mathcal{F}$ is *shifted*, i.e. $S_{i,j}(\mathcal{F}) = \mathcal{F}$ for all $1 \leq i < j \leq n$.
Let r denote the maximal integer for which there exists an $F \in \mathcal{F}$ of the form $F = \{i_1 < \cdots < i_{p-1} = r < i_p\}$.
Let $m = m_p(k)$. If $r \leq m-1$ then

$$|\mathcal{F}| \leq \binom{r}{p} + \binom{r}{p-1}(n-r) \leq \binom{m-1}{p} + \binom{m-1}{p-1}(n-m+1) =$$

$$\binom{m}{p} + \binom{m-1}{p-1}(n-m) \leq \binom{m}{p} + (k-m)(n-m) = b_p(n,k)\ .$$

If $r \geq k+1$ then $\mathcal{F}$ contains all sets $F = [p-2] \cup \{i,j\}$ for $p-1 \leq i < j \leq k+2$. In particular $F_i = [p-2] \cup \{p-1+2i, p+2i\} \in \mathcal{F}$ for $0 \leq i \leq t = \lfloor \frac{k-p}{2} \rfloor + 1$, hence $\lambda(\mathcal{F}) \geq |\bigcup_{i=0}^{t} F_i| \geq k+1$, a contradiction.
We shall therefore assume that $m \leq r \leq k$. For $j \in [n]$ let $\mathcal{F}(j) = \{F - \{j\} : j \in F \in \mathcal{F}\}$, $\deg(j) = |\mathcal{F}(j)|$. Since $\mathcal{F}$ is shifted $\deg(1) \geq \cdots \geq \deg(n)$. We consider two cases:
(i) If $\deg(j) \leq k+1-j$ for some $r+1 \leq j \leq k+1$ then

$$|\mathcal{F}| \leq \binom{r}{p} + \binom{r}{p-1}(j-r-1) + (k+1-j)(n-j+1) \leq$$

$$\max\{\binom{r}{p} + (k-r)(n-r), \binom{r}{p} + \binom{r}{p-1}(k-r)\}.$$

Since $u(x) = \binom{x}{p} + (n-x)(k-x)$ is convex for $x \geq p-1$, and $\binom{r}{p} + \binom{r}{p-1}(k-r) \leq \binom{k}{p}$, it follows that

$$|\mathcal{F}| \leq \max\{\binom{k}{p}, \binom{m}{p} + (n-m)(k-m)\} = \max\{a_p(k), b_p(n,k)\}\ .$$

(ii) If $\deg(j) \geq k+2-j$ for all $r+1 \leq j \leq k+1$, then we can successively choose $F_{k+1}, F_k, \ldots, F_{r+1} \in \mathcal{F}$ such that $j \in F_j$ and $\{F_j - \{j\}\}_{j=r+1}^{k+1}$ are all distinct. It follows that $\{F_j\}_{j=r+1}^{k+1}$ is a w-matching, and by shifting we may assume $\bigcup_{i=r+1}^{k+1} F_i = [t] \cup [r+1, k+1]$ for some $p-1 \leq t \leq r$. Since $[p-2] \bigcup \{r, r+1\} \in \mathcal{F}$ (by definition of r) it follows that
$C_j = [p-2] \cup \{r-2j-1, r-2j\} \in \mathcal{F}$ for $0 \leq j \leq \lfloor \frac{r-t}{2} \rfloor - 1 = b$.
Let $\mathcal{H} = \{F_j\}_{j=r+1}^{k+1} \cup \{C_j\}_{j=0}^{b}$, then $\mathcal{H} \subset \mathcal{F}$ is a weak matching and

$$\bigcup_{j=r+1}^{k+1} F_j \ \cup \ \bigcup_{j=0}^{b} C_j = \begin{cases} [k+1] & r-t \ \text{ even} \\ [k+1] - \{t+1\} & r-t \ \text{ odd} \end{cases} \tag{4}$$

Since $\lambda(\mathcal{F}) = k$ it follows that $r-t$ must be odd.
To bound $|\mathcal{F}|$ in this case, we need the following simple but somewhat technical observations:
Let $p-1 \leq i \leq t$, then $H_i = [p-2] \bigcup \{i, t+1\} \in \mathcal{F}$. Since $\bigcup_{H \in \mathcal{H}} H \bigcup H_i = [k+1]$ it follows that $\mathcal{H} \cup \{H_i\}$ is not a w-matching. Therefore there exists a $r+1 \leq j_i \leq k+1$ such that $|F_{j_i} \bigcap H_i| = p-1$, i.e. $F_{j_i} = [p-2] \cup \{i, j_i\}$. By shifting we may assume that $r+1 \leq j_t < j_{t-1} < \cdots < j_{p-1} \leq k+1$. Let $S = [p-2] \cup \{t+1, r+1\} \in \mathcal{F}$. If $j_t > r+1$, then $\mathcal{H} \cup \{S\}$ is a w-matching which covers $[k+1]$. Hence $j_t = r+1$. Denote $J = \{j_{p-1}, \ldots, j_t\} \subset [r+1, k+1]$, and for $p-1 \leq i \leq t$ let $F'_{j_i} = [p-2] \cup \{i+1, j_i\}$. Let

$$I = \{p-1 \leq i \leq t : j_i = r+t+1-i \ \text{ and } \ F'_{j_i} \in \mathcal{F}\}.$$

Let $l = \max([p-1, t] - I)$ if $I \neq [p-1, t]$ and $l = p-2$ otherwise. Since $t \in I$ it follows that $p-2 \leq l \leq t-1$. Let $G_j = F_j - \{j\}$, and let $\mathcal{G} = \{G_j\}_{j=r+1}^{k+1}$.

Claim 3.3
(a) *For any* $t+1 \leq j \leq n$, $\mathcal{F}(j) \cap \binom{[t]}{p-1} \subset \mathcal{G}$.
(b) *If* $[p-2] \not\subset G \in \mathcal{G}$ *then* $G \subset [l]$.
(c) *If* $F \in \mathcal{F} \cap \binom{[t]}{p}$ *and* $F \not\supset G$ *for all* $G \in \mathcal{G}$, *then* $F \subset [l]$.
(d) *If* $F \in \mathcal{F}$ *satisfies* $|F \cap [t+1, n]| \geq 2$, *then* $|F \cap [t+1, n]| = 2$ *and* $F \supset [p-2]$.
(e) $\mathcal{F}(j) \subset \{G_j\}_{j=r+t-l+1}^{k+1}$ *for all* $r+t-l+1 \leq j \leq n$
(f) $t-p+2 \leq k+1-r \leq t-l+\binom{l}{p-1}$.

Proof: We prove (a)-(e) by showing that in each case the assumption to the contrary implies the existence of a w-matching in $\mathcal{F}$ which covers $[k+1]$,

a contradiction.
(a) Otherwise there exists an $F \in \mathcal{F} \cap \binom{[t+1]}{p}$ such that $t+1 \in F$ and $F - \{t+1\} \notin \mathcal{G}$. It follows that $\mathcal{H} \cup \{F\}$ is a w-matching which covers $[k+1]$.
(b) Otherwise there exists a $j \in [r+1, k+1] - J$ such that $l+1 \leq \max G_j = i_0 \leq t$. It follows that $\mathcal{H} - \{F_{j_i}\}_{i=i_0}^t \cup \{F'_{j_i}\}_{i=i_0}^t$ is a w-matching which covers $[k+1]$.
(c) Otherwise let $l+1 \leq i_0 = \max F \leq t$, then $\mathcal{H} - \{F_{j_i}\}_{i=i_0}^t \cup \{F'_{j_i}\}_{i=i_0}^t \cup \{F\}$ is a w-matching which covers $[k+1]$.
(d) Otherwise F can be shifted to $F' \in \mathcal{F}$ such that $[p-2] \not\subset F' \in \mathcal{F} \cap \binom{[t+2]}{p}$ and $t+1, t+2 \in F'$. If $t \notin F'$ then $\mathcal{H} \cup \{F' - \{t+2\} \cup \{t\}\}$ is a w-matching covering $[k+1]$. Otherwise let $F'' = [p-2] \cup \{t+3, r+1\}$, then $\mathcal{H} - \{F_{r+1}, C_b\} \cup \{F', F''\}$ is a w-matching covering $[k+1]$.
(e) It suffices to show that $F = [p-2] \cup \{l+1, r+t-l+1\} \notin \mathcal{F}$. If $j_l = r+t-l+1$ then this follows from the definition of l. Otherwise $r+t-l+1 \notin J$ and $\mathcal{H} - \{F_{j_i}\}_{i=l+1}^t - \{F_{r+t-l+1}\} \cup \{F'_{j_i}\}_{i=l+1}^t \cup \{F\}$ is a w-matching covering $[k+1]$. Hence $F \notin \mathcal{F}$.
(f) The lefthand inequality is clear and the righthand inequality follows from (b).

□

Let $\mathcal{F}_1 = \mathcal{F} \cap \binom{[l]}{p}$, $\mathcal{F}_2 = \mathcal{F} \cap \binom{[t]}{p}$, $\mathcal{F}_3 = \mathcal{F} \cap \binom{[r]}{p}$, $\mathcal{F}_4 = \mathcal{F} \cap \binom{[r+t-l]}{p}$, $\mathcal{F}_5 = \mathcal{F}$. Let $\mathcal{F}'_1 = \mathcal{F}_1$ and $\mathcal{F}'_i = \mathcal{F}_i - \mathcal{F}_{i-1}$ for $2 \leq i \leq 5$.

Corollary 3.4
(1) $|\mathcal{F}'_1| \leq \binom{l}{p}$.
(2) $|\mathcal{F}'_2| \leq (t-l)((k+1-r)-(t-p+2)) + \binom{t-p+2}{2} - \binom{l-p+2}{2}$.
(3) $|\mathcal{F}'_3| \leq (r-t)(k+1-r) + \binom{r-t}{2}$.
(4) $|\mathcal{F}'_4| \leq (t-l)(k+1-r) + (t-l)(r-t)$.
(5) $|\mathcal{F}'_5| \leq (n-r-t+l)((k+1-r)-(t-l))$.

Proof: (2) follows from (b) and (c). Both (3) and (4) follow from (a) and (d). (5) follows from (e).

□

Applying Corollary 3.4 we obtain

$$|\mathcal{F}| = \sum_{i=1}^{5} |\mathcal{F}'_i| \leq \binom{l}{p} + (n-(r+t-l-1))(k-(r+t-l-1)) + (k+1-r)(r+t-2l-1) +$$

$$(t-l)(r+p-2t-1)+\binom{t-p+2}{2}-\binom{l-p+2}{2}+\binom{r-t}{2}\ .$$

Writing $s=r+t-l-1$ and using $k+1-r\leq(t-l)+\binom{l}{p-1}$ we obtain

$$|\mathcal{F}|\leq(n-s)(k-s)+\binom{l}{p}+\binom{l}{p-1}(s-l)+$$

$$(2r-2l-t+p-2)(t-l)+\binom{t-p+2}{2}-\binom{l-p+2}{2}+\binom{r-t}{2}=$$

$$(n-s)(k-s)+\binom{l}{p}+\binom{l}{p-1}(s-l)+\binom{s-l+1}{2}-\binom{t-l}{2}\ . \qquad (5)$$

Since $s-l\geq 2$ it follows that if $1\leq p-2\leq l-1$ and either $l\geq 3$ or $l=2$ and $s\geq 5$ then

$$\frac{\binom{s-l+1}{2}}{\binom{s-l}{2}}=\frac{s-l+1}{s-l-1}\leq l\leq\binom{l}{p-2}$$

hence

$$|\mathcal{F}|\leq(n-s)(k-s)+\binom{l}{p}+\binom{l}{p-1}(s-l)+\binom{l}{p-2}\binom{s-l}{2}\leq$$

$$(n-s)(k-s)+\binom{s}{p}\leq h_p(n,k)\ .$$

The remaining exceptional cases are:

1. $p=2$: Claim 3.3(f) implies $k+1-r=t$, hence $k=2t-1+(r-t)$ is even, contradicting our assumptions.

2. $p\geq 3$ and $l=p-2$: Again by Claim 3.3(f) $k+1-r=t-p+2$. Hence $s=k$ so by (5) $|\mathcal{F}|\leq\binom{k-p+3}{2}$. Since $k-p+3=r-t+2(t-p+2)$ must be odd it follows that $k\geq p+2$ and therefore $\binom{k-p+3}{2}\leq\binom{k}{p}$.

3. $l=2, s=4$ and $p=3$: It follows from Claim 3.3(f) that $k+1-r=t-1$. Therefore $k=r+t-2=s+1=5$, hence by (5) $|\mathcal{F}|\leq n+1<\max\{10,2n-5\}=h_3(n,5)$.

$\square$

We conclude with a problem concerning the uniqueness of the extremal cases. For a linear map $T\in\mathrm{End}(V)$ let $T^{\wedge p}\in\mathrm{End}(\bigwedge^p V)$ be given by $T^{\wedge p}(v_1\wedge\ldots\wedge v_p)=T(v_1)\wedge\ldots\wedge T(v_p)$.

Conjecture 3.5 *If* $W \subset \bigwedge^p V$ *satisfies* $\rho(W) \leq k$ *and* $\dim W = h_p(n,k)$ *then either* $W = T^{\wedge p}(\mathcal{A}_p(k))$ *or* $W = T^{\wedge p}(\mathcal{B}_p(n,k))$ *for some* $T \in \mathrm{GL}(V)$.

We have proved the conjecture when $k = n-1$ and p divides n.

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