Relative Leray Numbers via Spectral Sequences

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Dedicated to Michael O. Rabin, a trailblazing mathematician and computer scientist

Abstract

Let $\mathbb{F}$ be a fixed field and let X be a simplicial complex on the vertex set V. The Leray number $L(X;\mathbb{F})$ is the minimal d such that for all $i \geq d$ and $S \subset V$, the induced complex $X[S]$ satisfies $\tilde{H}_i(X[S];\mathbb{F}) = 0$. Leray numbers play a role in formulating and proving topological Helly type theorems. For two complexes X, Y on the same vertex set V, define the relative Leray number $L_Y(X;\mathbb{F})$ as the minimal d such that $\tilde{H}_i(X[V \setminus \tau];\mathbb{F}) = 0$ for all $i \geq d$ and $\tau \in Y$. In this paper we extend the topological colorful Helly theorem to the relative setting. Our main tool is a spectral sequence for the intersection of complexes indexed by a geometric lattice.

1 Introduction

Let $\mathbb{F}$ be a fixed field and let X be a simplicial complex on the vertex set V. All homology and cohomology appearing in the the sequel will be with $\mathbb{F}$ coefficients. The induced subcomplex of X on a subset $S \subset V$ is $X[S] = \{\sigma \in X : \sigma \subset S\}$.

Definition 1.1. *The* Leray number $L(X) = L(X;\mathbb{F})$ *of* X *over* $\mathbb{F}$ *is the minimal* d *such that* $\tilde{H}_i(X[S]) = 0$ *for all* $S \subset V$ *and* $i \geq d$. *The complex* X *is* d-Leray *over* $\mathbb{F}$ *if* $L(X) \leq d$.

First introduced by Wegner [18], the family $\mathcal{L}^d = \mathcal{L}^d_{\mathbb{F}}$ of d-Leray complexes over the field $\mathbb{F}$, has the following relevance to Helly type theorems. Let $\mathcal{F}$ be a family of sets. The *Helly number* $h(\mathcal{F})$ is the minimal positive integer h such that if a finite subfamily $\mathcal{G} \subset \mathcal{F}$ satisfies $\bigcap \mathcal{G}' \neq \emptyset$ for all $\mathcal{G}' \subset \mathcal{G}$ of cardinality $\leq h$, then $\bigcap \mathcal{G} \neq \emptyset$. Let $h(\mathcal{F}) = \infty$ if no such finite h exists. For example, Helly's classical theorem asserts that the Helly number of the family of convex sets in $\mathbb{R}^d$ is $d+1$. Helly type theorems can often be formulated as properties of the associated nerves. Recall that the *nerve* of a family of sets $\mathcal{F}$ is the simplicial complex $N(\mathcal{F})$ on the vertex set $\mathcal{F}$, whose simplices are all subfamilies $\mathcal{G} \subset \mathcal{F}$ such that $\bigcap \mathcal{G} \neq \emptyset$. A simple link between the Helly and Leray numbers is the inequality $h(\mathcal{F}) \leq L(N(\mathcal{F})) + 1$ (see e.g. (1.2) in [13]). A simplicial complex X is *d-representable* if $X = N(\mathcal{K})$ for a family $\mathcal{K}$ of convex sets in $\mathbb{R}^d$. Let $\mathcal{K}^d$ be the set of all d-representable complexes. Helly's theorem can then be stated as follows: If $X \in \mathcal{K}^d$ contains the full d-skeleton of its vertex set, then X is a simplex. The nerve lemma (see e.g. [5]) implies that $\mathcal{K}^d \subset \mathcal{L}^d$, but the latter family is much richer, and there is substantial interest in understanding to what extent Helly type statements for $\mathcal{K}^d$ remain true for $\mathcal{L}^d$. A basic example is the following. A finite family $\mathcal{F}$ of simplicial complexes in $\mathbb{R}^d$ is a *good*

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cover if for any $\mathcal{F}' \subset \mathcal{F}$, the intersection $\bigcap \mathcal{F}'$ is either empty or contractible. If $\mathcal{F}$ is a good cover in $\mathbb{R}^d$, then by the nerve lemma $N(\mathcal{F})$ is homotopic to $\bigcup \mathcal{F}$ and therefore $L(N(\mathcal{F})) \leq d$. Hence follows the Topological Helly's Theorem: If $\mathcal{F}$ is a good cover in $\mathbb{R}^d$, then $h(\mathcal{F}) \leq L(N(\mathcal{F})) + 1 \leq d + 1$.

The Colorful Helly Theorem due to Bárány and Lovász [2] is a fundamental result with a number of important applications in discrete geometry (see e.g. [7] and the references therein).

Theorem 1.2 ([2])**.** *Let $\mathcal{K}_1, \ldots, \mathcal{K}_{d+1}$ be $d+1$ finite families of convex sets in $\mathbb{R}^d$, such that $\bigcap_{i=1}^{d+1} K_i \neq \emptyset$ for all choices of $K_1 \in \mathcal{K}_1, \ldots, K_{d+1} \in \mathcal{K}_{d+1}$. Then there exists an $1 \leq i \leq d+1$ such that $\bigcap_{K \in \mathcal{K}_i} K \neq \emptyset$.*

In [11] we showed that the d-representability of $X = N(\bigcup_{i=1}^{d+1} \mathcal{K}_i)$ can be replaced by the weaker assumption that X is d-Leray.

Theorem 1.3 ([11])**.** *Let $V = \bigcup_{i=1}^{d+1} V_i$ be a partition of V, and let X be a d-Leray complex on V. View each V_i as a 0-dimensional complex and suppose that X contains the join $V_1 * \cdots * V_{d+1}$. Then there exists an $1 \leq i \leq d+1$ such that V_i is a simplex of X.*

In fact, the transversal matroid $V_1 * \cdots * V_{d+1}$ in the statement of Theorem 1.3, can be replaced by an arbitrary matroid. In the sequel we identify a matroid with the simplicial complex of its independent sets. We recall that every induced subcomplex $M[S]$ of a matroid M is pure, namely all maximal faces of $M[S]$ have the same dimension. This property can actually serve as the definition of a matroid in terms of the simplicial complex of its independent sets.

Theorem 1.4 ([11])**.** *Let M be a matroid with a rank function ρ_M, and let X be a d-Leray complex over some field $\mathbb{F}$, both on the same vertex set V. If $M \subset X$, then there exists a $\tau \in X$ such that $\rho_M(V \setminus \tau) \leq d$.*

For recent applications of Theorem 1.4 to graph theory and to combinatorial geometry see [1] and [8, 15, 9] respectively. While [1] uses Theorem 1.4 under the stronger assumption that X is d-collapsible, the results in [8, 15, 9] do require its full version for d-Leray complexes.

In this paper we prove a generalization of Theorem 1.4 using a new spectral sequence approach. Let X and Y be two complexes on the same vertex set V.

Definition 1.5. *The* relative Leray number *of X with respect to Y is*

$$L_Y(X) = L_Y(X; \mathbb{F}) = \min\{d : \tilde{H}_i(X[V \setminus \tau]) = 0 \ \textit{ for all } i \geq d \textit{ and } \tau \in Y\}.$$

Our main result is the following relative extension of Theorem 1.4. Let X, Y be simplicial complexes on the vertex set V. Let $Y^{\vee} = \{A \subset V : V \setminus A \notin Y\}$ denote the Alexander dual of Y as a subcomplex of the simplex on V.

Theorem 1.6. *Let M be a matroid such that $Y^{\vee} \subset M \subset X$. Then there exists a simplex $\tau \in X$ such that $\rho_M(V \setminus \tau) \leq L_Y(X)$.*

The paper is organized as follows. In section 2 we give a characterization of the relative Leray numbers in terms of links. In section 3 we construct a Mayer-Vietoris type spectral sequence (Proposition 3.1), and use it to establish a homological non-vanishing criterion (Corollary 3.4) for certain families of complexes indexed by a geometric lattice. This result, which may be of independent interest, is the main ingredient in the proof of Theorem 1.6 given in section 4.

2 Relative Leray Numbers via Links

We first recall a few definitions. Let X be a simplicial complex on the vertex set V. The star and link of a subset $\tau \subset V$ are given by

$$\begin{aligned} \mathrm{st}(X,\tau) &= \{\sigma \in X : \sigma \cup \tau \in X\}, \\ \mathrm{lk}(X,\tau) &= \{\sigma \in \mathrm{st}(X,\tau) : \sigma \cap \tau = \emptyset\}. \end{aligned}$$

Note that if $\tau \notin X$, then $\mathrm{st}(X,\tau) = \mathrm{lk}(X,\tau) = \{\,\}$ is the void complex. In particular, if $\tilde{H}_*(\mathrm{lk}(X,\tau)) \neq 0$ then $\tau \in X$. It is well known (see e.g. Proposition 3.1 in [12]) that $L(X) \leq d$ iff $\tilde{H}_i(\mathrm{lk}(X,\sigma)) = 0$ for all simplices $\sigma \in X$ and $i \geq d$. The relative version of this fact is the following

Proposition 2.1. *Let X, Y be complexes on the vertex set V. Then*

$$L_Y(X) = \tilde{L}_Y(X) := \min\{d : \tilde{H}_i(\mathrm{lk}(X,\sigma)) = 0 \text{ for all } i \geq d \text{ and } \sigma \in Y\}.$$

Proposition 2.1 is implicit in the proof of Claim 3.2 in [12], and can also be deduced from a result of Bayer, Charalambous and Popescu (see Theorem 2.8 in [3]). For completeness, we include a simple direct proof of a slightly stronger result, following the argument in [12].

Definition 2.2. *Let X be a complex on the vertex set V and let $A \subset V$. The pair (X, A) satisfies* property $P_d(k_1, k_2)$, *if $\tilde{H}_i\big(\mathrm{lk}\left(X[V \setminus \sigma_1], \sigma_2\right)\big) = 0$ for all $i \geq d$ and all disjoint $\sigma_1, \sigma_2 \subset A$ such that $|\sigma_1| \leq k_1$, $|\sigma_2| \leq k_2$.*

Proposition 2.3. *For a fixed pair (X, A) and $k_1 \geq 0$, $k_2 \geq 1$, the properties $P_d(k_1, k_2)$ and $P_d(k_1 + 1, k_2 - 1)$ are equivalent.*

Proof. Let τ_1, τ_2 be disjoint subsets of A such that $|\tau_1| \leq k_1 + 1$, $|\tau_2| \leq k_2 - 1$ and suppose $v \in \tau_1$. Let $\sigma_1 = \tau_1 \setminus \{v\}$, $\sigma_2 = \tau_2 \cup \{v\}$, and let Z be the complex on the vertex set $W = V \setminus (\sigma_1 \cup \tau_2)$ given by $Z = \mathrm{lk}(X[V \setminus \sigma_1], \tau_2)$. Write $Z = Z_1 \cup Z_2$ where

$$\begin{aligned} Z_1 &= Z[W \setminus \{v\}] = \mathrm{lk}(X[V \setminus \tau_1], \tau_2), \\ Z_2 &= \mathrm{st}(Z, v) = \mathrm{st}(\mathrm{lk}(X[V \setminus \sigma_1], \tau_2), v). \end{aligned} \tag{1}$$

Then

$$Z_1 \cap Z_2 = \mathrm{lk}(Z, v) = \mathrm{lk}\big(\mathrm{lk}\left(X[V \setminus \sigma_1], \tau_2\right), v\big) = \mathrm{lk}\left(X[V \setminus \sigma_1], \sigma_2\right). \tag{2}$$

As Z_2 is a cone and hence contractible, we obtain a Mayer-Vietoris exact sequence

$$\ldots \to \tilde{H}_{i+1}(Z) \to \tilde{H}_i\left(\mathrm{lk}(Z,v)\right) \to \tilde{H}_i\left(Z[W \setminus \{v\}]\right) \to \tilde{H}_i(Z) \to \ldots \quad . \tag{3}$$

Using the identifications of Z_1, Z_2 and $Z_1 \cap Z_2$ in (1) and (2), the exact sequence (3) takes the explicit form

$$\begin{aligned} &\ldots \to \tilde{H}_{i+1}\big(\mathrm{lk}(X[V \setminus \sigma_1], \tau_2)\big) \to \tilde{H}_i\big(\mathrm{lk}(X[V \setminus \sigma_1], \sigma_2)\big) \\ &\to \tilde{H}_i\big(\mathrm{lk}(X[V \setminus \tau_1], \tau_2)\big) \to \tilde{H}_i\big(\mathrm{lk}(X[V \setminus \sigma_1], \tau_2)\big) \to \ldots \quad . \end{aligned} \tag{4}$$

$\mathbf{P_d(k_1, k_2) \Rightarrow P_d(k_1 + 1, k_2 - 1)}$: Suppose (X, A) satisfies $P_d(k_1, k_2)$ and let $i \geq d$. Let τ_1, τ_2 be disjoint subsets of A such that $|\tau_1| = k_1 + 1$, $|\tau_2| \leq k_2 - 1$. Choose $v \in \tau_1$, and let $\sigma_1 = \tau_1 \setminus \{v\}$, $\sigma_2 = \tau_2 \cup \{v\}$. The assumption that $P_d(k_1, k_2)$ holds implies that the second and the fourth terms in

(4) vanish. It follows that $\tilde{H}_i\big(\mathrm{lk}(X[V \setminus \tau_1], \tau_2)\big) = 0$ as required.
$\mathbf{P_d(k_1+1, k_2-1) \Rightarrow P_d(k_1, k_2)}$: Suppose (X, A) satisfies $P_d(k_1+1, k_2-1)$ and let $i \geq d$. Let σ_1, σ_2 be disjoint subsets of A such that $|\sigma_1| \leq k_1, |\sigma_2| = k_2$. Choose $v \in \sigma_2$, and let $\tau_1 = \sigma_1 \cup \{v\}$ and $\tau_2 = \sigma_2 \setminus \{v\}$. The assumption that $P_d(k_1+1, k_2-1)$ holds implies that the first and the third terms in (4) vanish. It follows that $\tilde{H}_i\big(\mathrm{lk}(X[V \setminus \sigma_1], \sigma_2)\big) = 0$ as required.

□

Proof of Proposition 2.1. Clearly, $L_Y(X) = \max\{L_A(X) : A \in Y\}$, and $\tilde{L}_Y(X) = \max\{\tilde{L}_A(X) : A \in Y\}$. It therefore suffices to show that $L_A(X) = \tilde{L}_A(X)$ for a simplex A. Now, $L_A(X) \leq d$ iff (X, A) satisfies $P_d(|A|, 0)$, while $\tilde{L}_A(X) \leq d$ iff (X, A) satisfies $P_d(0, |A|)$. Finally, $P_d(|A|, 0)$ and $P_d(0, |A|)$ are equivalent by Proposition 2.3.

□

3 Empty Intersections and Non-Vanishing Homology

For a poset P and an element $x \in P$, let $P_{>x} = \{y \in P : y > x\}$ and $P_{\geq x} = \{y \in P : y \geq x\}$. Let $\Delta(P)$ denote the order complex of P, i.e. the simplicial complex on the vertex set P whose simplices are the chains $x_0 < \cdots < x_k$. Let M be a matroid with rank function ρ_M on the ground set V. Let $\mathcal{K}(M)$ denote the poset of all flats $K \neq V$ of M ordered by inclusion, and let $\mathcal{K}_0(M) = \{K \in \mathcal{K}(M) : \rho_M(K) > 0\}$. It is classically known (see e.g. [4]) that $\tilde{H}_j\left(\Delta\left(\mathcal{K}_0(M)\right)\right) = 0$ for $j \neq \rho_M(V) - 2$. Let $K \in \mathcal{K}(M)$ and let B_K be an arbitrary basis of K. The contraction of K from M is the matroid on $V \setminus K$ defined by $M/K = \{A \subset V \setminus K : B_K \cup A \in M\}$ (see e.g. [14]). The matroid M/K satisfies $\rho_{M/K}(V \setminus K) = \rho_M(V) - \rho_M(K)$ and $\mathcal{K}_0(M/K) \cong \mathcal{K}(M)_{>K}$. Let $\{Y_K : K \in \mathcal{K}(M)\}$ be a family of simplicial complexes such that $Y_K \cap Y_{K'} = Y_{K \cap K'}$ for all $K, K' \in \mathcal{K}(M)$. Let $Y = \bigcup_{K \in \mathcal{K}(M)} Y_K$. For $y \in Y$ let $K_y = \bigcap\{K \in \mathcal{K}(M) : y \in Y_K\} \in \mathcal{K}(M)$. The proof of the following result is an application of the method of simplicial resolutions (see e.g. Vassiliev's paper [17]).

Proposition 3.1. *There exists a first quadrant spectral sequence* $\{E^r_{p,q}\}$ *converging to* $H_*(Y)$ *whose* E^1 *term satisfies*

$$E^1_{p,q} \cong \bigoplus_{\substack{K \in \mathcal{K}(M) \\ \rho_M(K) = \rho_M(V) - p - 1}} H_q(Y_K) \otimes \tilde{H}_{p-1}\big(\Delta\left(\mathcal{K}_0(M/K)\right)\big). \tag{5}$$

Proof. Let $\rho_M(V) = m$. For $0 \leq p \leq m-1$ let

$$F_p = \bigcup_{\substack{K \in \mathcal{K}(M) \\ \rho_M(K) \geq m-p-1}} Y_K \times \Delta\big(\mathcal{K}(M)_{\geq K}\big).$$

Let $\varphi : F_{m-1} \to Y$ denote the projection on the first coordinate. For $y \in Y$, the fiber $\varphi^{-1}(y) = \{y\} \times \Delta\left(\mathcal{K}(M)_{\geq K_y}\right)$ is a cone on the vertex (y, K_y) and is therefore contractible. Hence, by the Vietoris-Begle theorem (see e.g. p. 344 in [16]), $H_*(F_{m-1}) \cong H_*(Y)$. The filtration $F_0 \subset \cdots \subset F_{m-1}$ thus gives rise to a spectral sequence $\{E^r_{p,q}\}$ that converges to $H_*(Y)$. Let

$$G_p = \bigcup_{\substack{K \in \mathcal{K}(M) \\ \rho_M(K) = m-p-1}} Y_K \times \Delta\big(\mathcal{K}(M)_{\geq K}\big).$$

Then $F_p = G_p \cup F_{p-1}$ and

$$G_p \cap F_{p-1} = \bigcup_{\substack{K \in \mathcal{K}(M) \\ \rho_M(K) = m-p-1}} Y_K \times \Delta\left(\mathcal{K}(M)_{>K}\right).$$

Additionally, if $K \neq K' \in \mathcal{K}(M)$ satisfy $\rho_M(K) = \rho_M(K') = m - p - 1$, then

$$\mathcal{K}(M)_{\geq K} \cap \mathcal{K}(M)_{\geq K'} \subset \mathcal{K}(M)_{>K}$$

and therefore

$$\Big(Y_K \times \Delta\left(\mathcal{K}(M)_{\geq K}\right)\Big) \cap \Big(Y_{K'} \times \Delta\left(\mathcal{K}(M)_{\geq K'}\right)\Big) \subset Y_K \times \Delta\left(\mathcal{K}(M)_{>K}\right). \tag{6}$$

We will also need the following simple

Claim 3.2. *Let $\{(S_\alpha, R_\alpha)\}_{\alpha \in A}$ be a finite family of pairs of cell complexes such that $R_\alpha \subset S_\alpha$ and $S_\alpha \cap S_{\alpha'} \subset R_\alpha$ for all $\alpha \neq \alpha' \in A$. Let $S = \cup_{\alpha \in A} S_\alpha$ and $R = \cup_{\alpha \in A} R_\alpha$. Then*

$$H_*(S, R) \cong \bigoplus_{\alpha \in A} H_*(S_\alpha, R_\alpha). \tag{7}$$

Proof. For $\alpha \in A$ let $i_\alpha : C_*(S_\alpha, R_\alpha) \to C_*(S, R)$ be the map induced by the inclusion $S_\alpha \to S$. Consider the surjection

$$\phi = \sum_{\alpha \in A} i_\alpha : \bigoplus_{\alpha \in A} C_*(S_\alpha, R_\alpha) \to C_*(S, R).$$

It is straightforward to check that the condition $S_\alpha \cap S_{\alpha'} \subset R_\alpha$ for $\alpha \neq \alpha'$ implies that ϕ injective. Hence ϕ is an isomorphism and (7) already holds at the chain level.

$\square$

We proceed with a computation of $E^1_{p,q}$, followed by a justification of each step.

$$\begin{aligned}
E^1_{p,q} &= H_{p+q}(F_p, F_{p-1}) \\
&\overset{(a)}{\cong} H_{p+q}(G_p, G_p \cap F_{p-1}) \\
&= H_{p+q}\Big(\bigcup_{\rho_M(K)=m-p-1} Y_K \times \Delta\left(\mathcal{K}(M)_{\geq K}\right), \bigcup_{\rho_M(K)=m-p-1} Y_K \times \Delta\left(\mathcal{K}(M)_{>K}\right)\Big) \\
&\overset{(b)}{\cong} \bigoplus_{\rho_M(K)=m-p-1} H_{p+q}\Big(Y_K \times \Delta\left(\mathcal{K}(M)_{\geq K}\right), Y_K \times \Delta\left(\mathcal{K}(M)_{>K}\right)\Big) \\
&\overset{(c)}{\cong} \bigoplus_{\rho_M(K)=m-p-1} \bigoplus_{i+j=p+q} H_i(Y_K) \otimes H_j\Big(\Delta\left(\mathcal{K}(M)_{\geq K}\right), \Delta\left(\mathcal{K}(M)_{>K}\right)\Big) \\
&\overset{(d)}{\cong} \bigoplus_{\rho_M(K)=m-p-1} \bigoplus_{i+j=p+q} H_i(Y_K) \otimes \tilde{H}_{j-1}\big(\Delta\left(\mathcal{K}(M)_{>K}\right)\big) \\
&\overset{(e)}{\cong} \bigoplus_{\rho_M(K)=m-p-1} \bigoplus_{i+j=p+q} H_i(Y_K) \otimes \tilde{H}_{j-1}\big(\Delta\left(\mathcal{K}_0(M/K)\right)\big).
\end{aligned} \tag{8}$$

Details of the derivation: (a) is the excision isomorphism, (b) is a consequence of (6) and Claim 3.2, (c) is the Künneth formula, (d) follows from the the contractibility of $\Delta(\mathcal{K}(M)_{\geq K})$ and the exact sequence of the pair $\Big(\Delta(\mathcal{K}(M)_{\geq K}), \Delta(\mathcal{K}(M)_{>K})\Big)$, and finally (e) follows from the poset isomorphism $\mathcal{K}(M)_{>K} \cong \mathcal{K}_0(M/K)$ given by $L \to L \setminus K$.

If $\rho_M(K) = m - p - 1$ then $\tilde{H}_{j-1}\big(\Delta(\mathcal{K}_0(M/K))\big) = 0$ for

$$j - 1 \neq \rho_{M/K}(V \setminus K) - 2 = \rho_M(V) - \rho_M(K) - 2 = p - 1.$$

It thus follows from (8) that

$$E^1_{p,q} \cong \bigoplus_{\rho_M(K)=m-p-1} H_q(Y_K) \otimes \tilde{H}_{p-1}\big(\Delta(\mathcal{K}_0(M/K))\big).$$

□

For the proof of the next result, we recall the combinatorial version of the Alexander duality theorem, see e.g. section 6 in [10] and Theorem 2 in [6]. For a simple direct proof see section 2 in [3].

Theorem 3.3 (Combinatorial Alexander Duality)**.** *Let X be a simplicial complex on V. Then for all $0 \leq q \leq |V| - 1$*

$$\tilde{H}_{|V|-2-q}(X; \mathbb{Z}) \cong \tilde{H}^{q-1}(X^\vee; \mathbb{Z}).$$

In particular, $\tilde{H}_{|V|-2-q}(X; \mathbb{F}) \cong \tilde{H}_{q-1}(X^\vee; \mathbb{F})$ for homology with field coefficients $\mathbb{F}$.

Let $\{Z_K : Z \in \mathcal{K}(M)\}$ be a family of complexes such that $Z_K \cup Z_{K'} = Z_{K \cap K'}$ for all $K, K' \in \mathcal{K}(M)$. Proposition 3.1 implies the following

Corollary 3.4. *Suppose that $\bigcap_{K \in \mathcal{K}(M)} Z_K = \{\emptyset\}$. Then there exist $0 \leq p \leq \rho_M(V) - 1$ and $K \in \mathcal{K}(M)$ of rank $\rho_M(K) = \rho_M(V) - p - 1$, such that $\tilde{H}_{p-1}(Z_K) \neq 0$.*

Proof. Let $\rho_M(V) = m$. We may assume that all the Z_K's are subcomplexes of the $(N-1)$-dimensional simplex Δ_{N-1} where $N > m$. Let $Y_K = Z_K^\vee$ be the Alexander dual of Z_K in Δ_{N-1}. Then for all $K, K' \in \mathcal{K}(M)$

$$Y_K \cap Y_{K'} = Z_K^\vee \cap Z_{K'}^\vee = (Z_K \cup Z_{K'})^\vee = Z_{K \cap K'}^\vee = Y_{K \cap K'}.$$

Moreover,

$$Y = \bigcup_{K \in \mathcal{K}(M)} Y_K = \bigcup_{K \in \mathcal{K}(M)} Z_K^\vee = \left(\bigcap_{K \in \mathcal{K}(M)} Z_K \right)^\vee = \{\emptyset\}^\vee = \partial \Delta_{N-1} \cong S^{N-2}.$$

By Proposition 3.1 there exists a spectral sequence $\{E^r_{p,q}\}$ converging to $H_*(Y) = H_*(S^{N-2})$ with E^1 terms given by (5). As $H_{N-2}(S^{N-2}) \neq 0$, it follows that there exist $0 \leq p \leq m-1$ and $q \geq 0$ such that $p + q = N - 2$ and $E^1_{p,q} \neq 0$. Hence, by (5), there exists a flat $K \in \mathcal{K}(M)$ of rank $\rho_M(K) = m - p - 1$ such that $H_q(Y_K) \neq 0$. Note that $q = N - 2 - p > m - 2 - p \geq 0$. By Alexander duality we obtain

$$0 \neq H_q(Y_K) = \tilde{H}_q(Y_K) = \tilde{H}_q(Z_K^\vee) \cong \tilde{H}_{N-3-q}(Z_K) = \tilde{H}_{p-1}(Z_K).$$

□

4 A Relative Topological Colorful Helly Theorem

Proof of Theorem 1.6. It suffices to show the existence of a subset $\tau \subset V$ such that

$$\tau \in X \cap Y \qquad \& \qquad \tilde{H}_{\rho_M(V\setminus\tau)-1}\big(\mathrm{lk}(X,\tau)\big) \neq 0. \tag{9}$$

Indeed, if (9) holds then of course $\tau \in X$. Moreover, as $\tau \in Y$ and $\tilde{H}_{\rho_M(V\setminus\tau)-1}\big(\mathrm{lk}(X,\tau)\big) \neq 0$, it follows from Proposition 2.1 that $\rho_M(V \setminus \tau) \leq L_Y(X)$, as required. The idea behind finding such τ is that the assumption $Y^\vee \subset M \subset X$ gives rise to a naturally defined family of complexes $\{Z_K\}$ indexed by the flats of the dual matroid M^* (see below), whose intersection is empty. Corollary 3.4 then implies the existence of a $p \geq 0$ and a flat K of M^*, such that $\rho_{M^*}(K) = \rho_{M^*}(V) - p - 1$ and $\tilde{H}_{p-1}(Z_K) \neq 0$. Finally, it can be shown that these two conditions imply (9) for $\tau = K$.

We proceed with the detailed argument. Let $M^* = \{\sigma \subset V : \rho_M(V\setminus\sigma) = \rho_M(V)\}$ be the dual matroid of M. The rank function of M^* satisfies $\rho_{M^*}(A) = |A| - \rho_M(V) + \rho_M(V \setminus A)$. For $K \in \mathcal{K}(M^*)$, we view the simplices of $X^\vee \setminus X^\vee[K]$ as a poset ordered by inclusion, and consider its order complex $Z_K = \Delta\left(X^\vee \setminus X^\vee[K]\right)$. Then

$$Z_K \cup Z_{K'} = \Delta\left(X^\vee \setminus X^\vee[K]\right) \cup \Delta\left(X^\vee \setminus X^\vee[K']\right) = \Delta\left(X^\vee \setminus X^\vee[K \cap K']\right) = Z_{K\cap K'}.$$

Let $\mathrm{sd}\left(X^\vee[V \setminus K]\right) = \Delta\left(X^\vee[V \setminus K] \setminus \{\emptyset\}\right)$ denote the barycentric subdivision of $X^\vee[V \setminus K]$. The inclusion map

$$\mathrm{sd}\left(X^\vee[V \setminus K]\right) \to \Delta\left(X^\vee \setminus X^\vee[K]\right) = Z_K$$

is a homotopy equivalence. Indeed, the retraction $\Delta\left(X^\vee \setminus X^\vee[K]\right) \to \mathrm{sd}\left(X^\vee[V \setminus K]\right)$ is given by the simplicial map that sends a vertex σ of $\Delta\left(X^\vee \setminus X^\vee[K]\right)$ to the vertex $\sigma \setminus K$ of $\mathrm{sd}\left(X^\vee[V \setminus K]\right)$. It follows that there is a homotopy equivalence

$$Z_K \simeq \mathrm{sd}\left(X^\vee[V \setminus K]\right) \simeq X^\vee[V \setminus K]. \tag{10}$$

Let $\sigma \in X^\vee$. Then $V \setminus \sigma \notin X$, and hence $V \setminus \sigma \notin M$. Therefore σ does not contain a basis of M^*, and thus $\sigma \subset K$ for some $K \in \mathcal{K}(M^*)$. Hence σ is not a vertex of Z_K. It follows that

$$\bigcap_{K \in \mathcal{K}(M^*)} Z_K = \{\emptyset\}.$$

By Corollary 3.4 there exist $0 \leq p \leq \rho_{M^*}(V) - 1$ and $K \in \mathcal{K}(M^*)$ such that

$$\tilde{H}_{p-1}(Z_K) \neq 0 \tag{11}$$

and

$$\rho_{M^*}(K) = \rho_{M^*}(V) - p - 1. \tag{12}$$

As $K \in \mathcal{K}(M^*)$, it follows that $V \setminus K \notin M$. The assumption $Y^\vee \subset M$ then implies that $V \setminus K \notin Y^\vee$, hence $K \in Y$. Furthermore, (12) is equivalent to

$$\rho_M(V \setminus K) = |V| - |K| - p - 1. \tag{13}$$

Note that

$$X^\vee[V \setminus K] = \mathrm{lk}(X, K)^\vee. \tag{14}$$

Indeed,

$$\begin{aligned} \sigma \in \mathrm{lk}(X,K)^\vee &\iff \sigma \subset V \setminus K \;\; \& \;\; (V \setminus K) \setminus \sigma \notin \mathrm{lk}(X,K) \\ &\iff \sigma \subset V \setminus K \;\; \& \;\; V \setminus \sigma \notin X \iff \sigma \in X^\vee[V \setminus K]. \end{aligned}$$

Using (11),(10),(14), Alexander duality and (13), we obtain

$$\begin{aligned} 0 \neq \tilde{H}_{p-1}(Z_K) &\cong \tilde{H}_{p-1}\left(X^\vee[V \setminus K]\right) = \tilde{H}_{p-1}\left(\mathrm{lk}(X,K)^\vee\right) \\ &\cong \tilde{H}_{|V|-|K|-p-2}\big(\mathrm{lk}(X,K)\big) = \tilde{H}_{\rho_M(V \setminus K)-1}\big(\mathrm{lk}(X,K)\big). \end{aligned}$$

As $K \in Y$, it follows from Proposition 2.1 that $\rho_M(V \setminus K) \leq L_Y(X)$. Finally, $K \in X$ since $\tilde{H}_* \left(\mathrm{lk}(X,K)\right) \neq 0$.

□

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