

Exam in Algebraic Topology 1.2.22

This is a closed-book exam. Please write clearly and answer all five questions.

Notation: $S^n = \{x \in \mathbb{R}^{n+1} : |x| = 1\}$ denotes the n -dimensional unit sphere in \mathbb{R}^{n+1} .

For two simplicial complexes K and L and a continuous map $f : |K| \rightarrow |L|$, denote by f_* the induced map $f_* : H_k(K) \rightarrow H_k(L)$.

The order complex $\Delta(P)$ of a poset (P, \leq) is the simplicial complex on the vertex set P whose simplices are all increasing chains of P .

For an oriented k -simplex σ in a simplicial complex K , denote by σ^* the k -cochain that satisfies $\sigma^*(\sigma) = 1$, and $\sigma^*(\tau) = 0$ for any oriented k -simplex τ that is not a permutation of σ .

For $i = 1, 2$ let X_i be a simplicial complex on the vertex set V_i . Assume that $V_1 \cap V_2 = \emptyset$. The join $X_1 * X_2$ is the simplicial complex on the vertex set $V_1 \cup V_2$ whose simplices are all unions $\sigma_1 \cup \sigma_2$ where $\sigma_1 \in X_1, \sigma_2 \in X_2$. Note: If X_2 is a cone, then so is $X_1 * X_2$.

1. (a) (5 points) Let Y_1, Y_2 be two subcomplexes of Y such that $Y = Y_1 \cup Y_2$. Write down the Mayer-Vietoris exact sequence associated with the pair (Y_1, Y_2) .
(b) (15 points) Let X be a simplicial complex, and let $\partial\Delta_2$ be the boundary complex of the 2-simplex (i.e. $\partial\Delta_2$ is the boundary of a triangle). Assume that the vertex sets of X and Δ_2 are disjoint. Compute the homology of the join $X * \partial\Delta_2$ in terms of the homology of X .
2. Let $0 \leq k < n$. Let $A_{n,k} = \{(x_0, \dots, x_n) \in S^n : x_{k+1} = \dots = x_n = 0\}$. Note that A is a k -dimensional sphere. Let $X_{n,k}$ be the space obtained from S^n by identifying all points of $A_{n,k}$ to one point (i.e. $X_{n,k}$ is the quotient space S^n / \sim , where $x \sim y$ for all $x, y \in A_{n,k}$, and $x \not\sim y$ for all x, y such that $x \neq y$ and $\{x, y\} \not\subset A_{n,k}$).
(a) (4 points) Let $Z \subset Y$ be simplicial complexes. Write down the long exact sequence for the pair (Y, Z) .
(b) (4 points) Draw (roughly) $X_{1,0}, X_{2,0}, X_{2,1}$.
(c) (12 points) Compute the homology of $X_{n,k}$ for all $0 \leq k < n$.
3. (a) (5 points) Define the notion of a simplicial n -dimensional homology manifold.
(b) (15 points) Suppose that K is a connected simplicial n -dimensional homology manifold. Prove that for any two n -dimensional simplices $\sigma, \tau \in K$, there exists a sequence of n -dimensional simplices $\sigma_1, \dots, \sigma_m$ such that $\sigma_1 = \sigma, \sigma_m = \tau$ and $|\sigma_i \cap \sigma_{i-1}| = n$ for all $2 \leq i \leq m$.

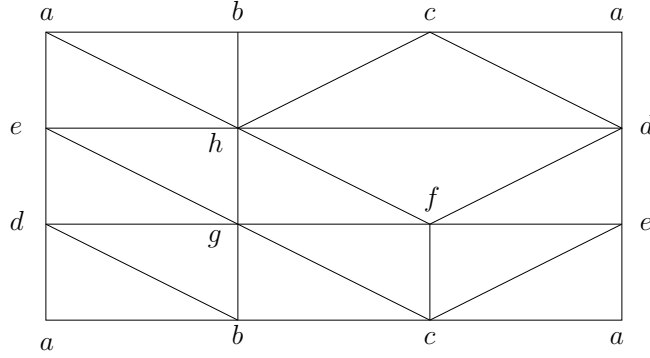


Figure 1: A 2-dimensional complex K

4. Let K denote the 2-dimensional complex depicted above. Let σ be the oriented 2-simplex $\sigma = [d, e, f] \in K$.
 - (a) (5 points) Find a block dissection of K with one 2-dimensional block, two 1-dimensional blocks and one 0-dimensional block.
 - (b) (5 points) Compute the homology of K with integer coefficients using the block dissection from (a).
 - (c) (5 points) Does there exist a $u \in C^1(K; \mathbb{Z})$ whose 1-coboundary $\delta_1 u$ satisfies $\delta_1 u = \sigma^*$? If your answer is positive, find such u .
 - (d) (5 points) Does there exist a $v \in C^1(K; \mathbb{Z})$ whose 1-coboundary $\delta_1 v$ satisfies $\delta_1 v = 2\sigma^*$? If your answer is positive, find such v .
5.
 - (a) (5 points) Formulate the Acyclic Carrier Theorem.
 - (b) (5 points) Let (P, \leq_P) , (Q, \leq_Q) be two posets and let $f, g : P \rightarrow Q$ be two nondecreasing maps, i.e. $f(x) \leq f(y)$, $g(x) \leq g(y)$ for every $x \leq y \in P$. Prove that if $f(x) \leq g(x)$ for all $x \in P$, then $f_* = g_*$.
 - (c) (10 points) Suppose (L, \leq) is a lattice, i.e. for any two elements $x, y \in L$ there exist elements $x \wedge y, x \vee y \in L$ such that $x \wedge y$ is the unique maximal element of $\{z \in L : z \leq x, z \leq y\}$ and $x \vee y$ is the unique minimal element in $\{z \in L : z \geq x, z \geq y\}$. Any lattice has unique minimal element $\hat{0}$ and a unique maximal element $\hat{1}$. Let $\hat{L} = L \setminus \{\hat{0}, \hat{1}\}$. The minimal elements of \hat{L} are called the atoms of L , and the maximal elements of \hat{L} are called the coatoms of L . Prove that if there exists an atom $x \in L$ such that $x \leq y$ for every coatom y of L , then $\Delta(\hat{L})$ is acyclic.