

$$1.b. \text{ Let } Y = X + \partial\Delta_2 = X + {}_2\Delta_3, \quad Y_1 = X + {}_2\Delta_3, \quad Y_2 = X + \overline{{}_2\Delta_3}$$

then $Y = Y_1 \cup Y_2, \quad Y_1 \cap Y_2 = X + {}_{2,3} = \Sigma X.$

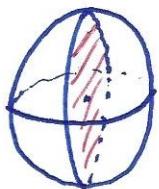
Both Y_1, Y_2 are cones over the vertex 1, hence $\tilde{H}_i(Y_1) = \tilde{H}_i(Y_2) = 0.$

By Mayer-Vietoris

$$0 = \tilde{H}_i(Y_1) \oplus \tilde{H}_i(Y_2) \rightarrow \tilde{H}_i(Y) \rightarrow \tilde{H}_{i-1}(Y_1 \cap Y_2) = \tilde{H}_{i-1}(\Sigma X) \rightarrow \tilde{H}_{i-1}(Y_1) \oplus \tilde{H}_{i-1}(Y_2) = 0$$

$$\Rightarrow \tilde{H}_i(Y) \cong \tilde{H}_{i-1}(\Sigma X) \cong \tilde{H}_{i-2}(X).$$

2.c. Clearly $X_{n,k} \cong S^n \cup \text{cone}(A_{n,k})$, hence



$$\tilde{H}_i(X_{n,k}) \cong H_i(S^n, A_{n,k}) \cong H_i(S^n, S^k).$$

Consider the long exact sequence for (S^n, S^k) :

$$(*) \quad \tilde{H}_i(S^k) \rightarrow \tilde{H}_i(S^n) \rightarrow H_i(S^n, S^k) \rightarrow \tilde{H}_{i-1}(S^k) \rightarrow \tilde{H}_{i-1}(S^n)$$

If $i > n+k+1$ then $\tilde{H}_i(S^n) = \tilde{H}_{i-1}(S^k) = 0$, hence $H_i(S^n, S^k) = 0$

If $i = n+k+1$ then $\tilde{H}_i(S^k) = \tilde{H}_{i-1}(S^n) = 0$, $\tilde{H}_i(S^n) \cong \tilde{H}_{i-1}(S^k) \cong \mathbb{Z}$

hence by $(*)$: $H_n(S^n, S^{n-1}) \cong \mathbb{Z} \oplus \mathbb{Z}.$

If $i = k+1 < n$, then $\tilde{H}_i(S^n) = \tilde{H}_{i-1}(S^n) = 0$, hence by $(*)$:

$$H_{k+1}(S^n, S^k) \cong \tilde{H}_k(S^k) \cong \mathbb{Z}$$

If $i = n > k+1$ then $\tilde{H}_n(S^k) = \tilde{H}_{n-1}(S^k) = 0$, hence by $(*)$

$$H_n(S^n, S^k) \cong \tilde{H}_n(S^n) \cong \mathbb{Z}.$$

Thus

$$\tilde{H}_i(X_{n,k}) \cong H_i(S^n, S^k) \cong \begin{cases} \mathbb{Z} & i = n = k+1 \\ \mathbb{Z} & i = k+1 < n \\ \mathbb{Z} & i = n > k+1 \\ 0 & \text{otherwise} \end{cases}$$

3.b. Let $K(n)$ denote the family of n -simplices of K .

As K is an n -dimensional homology manifold, it follows that

$\bar{z} = \sum_{\sigma \in K(n)} \sigma \in H_n(K)$. Suppose for contradiction that the conclusion

does not hold. Then there is a partition $K(n) = S_1 \cup S_2$ $S_1, S_2 \neq \emptyset$,

such that $|\sigma_1 \cap \sigma_2| \leq n-1 \wedge \sigma_1 \in S_1, \sigma_2 \in S_2$. Then, letting $z_i = \sum_{\sigma \in S_i}$

$$0 = \partial_n \bar{z} = \partial_n(z_1 + z_2) = \partial_n z_1 + \partial_n z_2 \Rightarrow \partial_n z_i = 0 \quad i=1,2$$

$\Rightarrow \dim_{\mathbb{Z}_2} H_n(K; \mathbb{Z}_2) \geq 2$, in contradiction with Poincaré duality

$$H_n(K; \mathbb{Z}_2) \cong H_0(K; \mathbb{Z}_2) \cong \mathbb{Z}_2$$

By connectedness of K

4. a. $e_0 = \{a\}, \dot{e}_0 = \emptyset, e_{1,1} = \{ab, bc, ca\}, e_{1,2} = \{ae, ed, da\},$

$$\dot{e}_{1,1} = \dot{e}_{1,2} = \dot{e}_0, \quad e_2 = K, \quad \dot{e}_2 = e_{1,1} \cup e_{1,2}.$$

b. Let u_2 = sum of all 2-simplices of K , oriented
anticlockwise be a generator of $H_2(e_2, \dot{e}_2)$.

Let $u_{1,1} = [a,c] + [c,b] + [b,a]$ be a generator of $H_1(e_{1,1}, \dot{e}_{1,1})$

Let $u_{1,2} = [a,e] + [e,d] + [d,a]$ " " " " $H_1(e_{1,2}, \dot{e}_{1,2})$

Let $u_0 = [a]$ be a generator of $H_0(e_0, \dot{e}_0)$

$$0 \rightarrow H_2(e_2, \dot{e}_2) \xrightarrow{d_2} H_1(e_{1,1}, \dot{e}_{1,1}) \oplus H_1(e_{1,2}, \dot{e}_{1,2}) \xrightarrow{d_1} H_0(e_0, \dot{e}_0) \rightarrow 0$$

$$0 \rightarrow \langle u_2 \rangle \rightarrow \langle u_{1,1} \rangle \oplus \langle u_{1,2} \rangle \rightarrow \langle u_0 \rangle \rightarrow 0$$

Then $d_2 d_2 = 2 d_{1,2}$, $d_1 d_{1,1} = d_1 d_{1,2} = 0$. Hence

$$k \rightarrow (0, \omega_k) \quad (\omega, l) \rightarrow 0$$

$$0 \rightarrow \mathbb{Z} \rightarrow \mathbb{Z} \oplus \mathbb{Z} \rightarrow \mathbb{Z} \rightarrow 0$$

$$\Rightarrow H_2(k) = \ker d_2 = 0, \quad H_1(k) \cong \frac{\mathbb{Z} \oplus \mathbb{Z}}{\text{Im } d_2} \cong \mathbb{Z} \oplus \mathbb{Z}/2, \quad H_0(k) \cong \mathbb{Z}$$

c. If w is an oriented edge of K , say $w = [x, y]$

then $\delta_1 w^* = \alpha^* + \beta^*$ where $\alpha = [x, y, z_1]$ $\beta = [x, y, z_2]$

where $(x, y, z_1), (x, y, z_2)$ are the two 2-simplexes that contain (x, y) .

It follows that for all $u \in C^1(K; \mathbb{Z})$, $\delta_1 u^*$ is a sum $\sum_{i=1}^L \tau_i^*$ where L is even. In particular $\delta_1 u + r^*$.

d. Let $v = \cancel{[d, f]^*} + [d, f]^* + [h, f]^* + [h, g]^* + [e, g]^* + [e, d]^*$

Then $\delta_1^* v = -2 \alpha^*$

5.b. For $\sigma = \{x_0 < \dots < x_k\} \in \Delta(P)$ define

$L(\sigma) = \Delta(\{f(x_0), \dots, f(x_k), g(x_0), \dots, g(x_k)\})$. $L(\sigma)$ is a cone on $f(x_0)$, hence contractible, hence acyclic. Clearly $f(\sigma), g(\sigma) \subset L(\sigma)$ and therefore $f_x = g_x$

c. Define $f, g, h : \hat{\mathbb{L}} \rightarrow \hat{\mathbb{L}}$ by $f(y) = x$, $g(y) = x \vee y$, $h(y) = y$.

f, g, h are clearly monotone and $f(y) \leq g(y) \geq h(y)$ for all $y \in \hat{\mathbb{L}}$. It follows that $f_* = g_* = h_*$.

$$\begin{aligned} \text{Therefore } \quad & H_i(\Delta(\hat{\mathbb{L}})) = h_*(H_i(\Delta(\hat{\mathbb{L}}))) = f_*(H_i(\Delta(\hat{\mathbb{L}}))) \\ & = H_i(\{x\}) \end{aligned}$$

i.e. $\Delta(\hat{\mathbb{L}})$ is acyclic.