Near Coverings and Cosystolic Expansion

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Abstract

Let X, Y be simplicial complexes and let $f : Y \to X$ be a simplicial surjective map. We introduce a notion of deficiency of f, denoted by $m_f(Y)$, that measures the average local failure of $f : Y \to X$ to be a covering map. We show, roughly speaking, that if $m_f(Y)$ is small and and if the non-abelian cosystolic expansion of X is large, then f is close to a genuine covering map. Our main result is a lower bound on the 1-cosystolic expansion with G coefficients of geometric lattices, with an application to near coverings of the 2-dimensional spherical building $A_3(\mathbb{F}_q)$.

1 Introduction

Many central topological structures, e.g. vector bundles and covering spaces, are defined in terms of local conditions. Classification theorems for such structures are often formulated in cohomological terms. For example, real line bundles over a compact space X are classified by $H^1(X;\mathbb{Z}_2)$, while complex line bundles over X are classified by $H^2(X;\mathbb{Z})$ (see, e.g., [17]). A natural challenge that arises is to formulate and prove approximate (or stability) versions of such classification theorems. Roughly speaking, such results would state that under suitable assumptions on X, if a structure satisfies all but a small fraction of the local conditions, then it corresponds to a cochain that is close (in an appropriate sense) to a cocycle.

In this paper we establish a stability version of the well known classification of G-covering spaces of a complex X by the first cohomology set $H^1(X;G)$, with an application to near coverings of the 2-dimensional spherical building $A_3(\mathbb{F}_q)$. Our first result (Theorem 1.7) provides a connection between the average local deviation of a map from a covering map, its proximity (again, in a precise sense defined below) to a genuine covering map, and its first expansion constant. The main technical result of this paper (Theorem 3.1) gives a lower bound on the 1-expansion of the geometric lattices over an arbitrary group G, with an application to expansion and cover-stability of the spherical building $A_3(\mathbb{F}_q)$.

In the following three subsections we describe the combinatorial and topological ingredients that appear in the formulation of our results. In Subsection 1.1 we define the deficiency of a simplicial map, a notion that will serve as a measure of the local failure of the map to be a covering map. Subsection 1.2 is concerned with the 1-cohomology $H^1(X;G)$ of a complex X over a finite coefficient group G, and its relation to G-coverings of X. A key element in this work is a notion of high-dimensional expansion that came up independently in the study of random simplicial complexes [9, 14], and in Gromov's remarkable work on the topological overlap property [7]. In Subsection 1.3 we recall the definition of the specific expansion constant needed here, namely the cosystolic 1-expansion of X with non-abelian coefficients $h_1(X;G)$. Finally, in Subsection 1.4 we state our results.

1.1 Deficiency of a Simplicial Surjection

Let X be an $(n-1)$-dimensional pure simplicial complex on the vertex set V. Let $X(k)$ denote the set of k-simplices of X, and let $X_{\mathrm{ord}}(k)$ denote the set of ordered k-simplices of X. Let $f_k(X) = |X(k)|$. The

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star and the *link* of a simplex $\tau \in X$ are given by

$$\begin{aligned}\mathrm{st}(X,\tau) &= \{\sigma \in X : \sigma \cup \tau \in X\},\\ \mathrm{lk}(X,\tau) &= \{\sigma \in \mathrm{st}(X,\tau) : \sigma \cap \tau = \emptyset\}.\end{aligned}$$

Following [6], define a weight function c_X on the simplices of X by

$$c_X(\sigma) = \frac{|\{\tau \in X(n-1) : \tau \supset \sigma\}|}{\binom{n}{|\sigma|} f_{n-1}(X)} = \frac{f_{n-|\sigma|-1}(\mathrm{lk}(X,\sigma))}{\binom{n}{|\sigma|} f_{n-1}(X)}.$$

Note that $\sum_{\sigma\in X(k)} c_X(\sigma) = 1$ for $0 \le k \le n-1$, and that if $\tau_1 \in X$ and $\tau_2 \in \mathrm{lk}(X,\tau_1)$ then

$$c_X(\tau_1)c_{\mathrm{lk}(X,\tau_1)}(\tau_2) = \binom{|\tau_1|+|\tau_2|}{|\tau_1|}^{-1} c_X(\tau_1 \cup \tau_2).$$

In particular, if $v \in X(0)$ and $e \in \mathrm{lk}(X,v)(1)$ then

$$c_X(v) \cdot c_{\mathrm{lk}(X,v)}(e) = \frac{1}{3} \cdot c_X(v \cup e). \tag{1}$$

Let Y be another simplicial complex and let $p : Y \to X$ be a surjective simplicial map. The pair (Y,p) is a *covering* of X if for any $u \in X(0)$ and $\tilde{u} \in p^{-1}(u)$, the induced mapping $p : \mathrm{st}(Y,\tilde{u}) \to \mathrm{st}(X,u)$ is an isomorphism. Note that in this case, the associated mapping $p : |Y| \to |X|$ between the geometric realizations of Y and X, is a covering in the usual topological sense as well. Consider now an arbitrary surjective simplicial map $f : Y \to X$ between two pure simplicial complexes Y and X. For a vertex $\tilde{u}$ of Y with an image $f(\tilde{u}) = u$, let

$$D_f(\tilde{u}) = \{e \in \mathrm{lk}(X,u)(1) : e \notin f(\mathrm{lk}(Y,\tilde{u}))\}.$$

Define the *local deficiency* of f at $\tilde{u}$ by

$$\mu_f(\tilde{u}) = \sum_{e\in D_f(\tilde{u})} c_{\mathrm{lk}(X,u)}(e).$$

The *deficiency* of the map $f : Y \to X$ is a weighted average of $\mu_f(\tilde{u})$ over all $\tilde{u} \in Y(0)$ given by

$$m_f(Y) = \sum_{u\in X(0)} \frac{c_X(u)}{|f^{-1}(u)|} \sum_{\tilde{u}\in f^{-1}(u)} \mu_f(\tilde{u}).$$

Remark 1.1. *If f is a covering map, then clearly $m_f(Y) = 0$. In the sequel - see Definition 1.3 - we will confine the discussion to maps $f : Y_\phi \to X$ where ϕ is a G-valued 1-cochain of X. For such maps, $m_f(Y_\phi) = 0$ iff f is a covering map, and $m_f(Y_\phi)$ may be viewed as a measure of the failure of f to be a covering map, see Remark 1.4(ii).*

1.2 Non-Abelian First Cohomology and Covering Maps

Let X be a finite simplicial complex and let G be a finite multiplicative group. Let $C^0(X;G)$ denote the group of G-valued functions on $X(0)$ with pointwise multiplication, and let

$$C^1(X;G) = \left\{\phi : X_{\mathrm{ord}}(1) \to G : \phi(u,v) = \phi(v,u)^{-1}\right\}.$$

The 0-coboundary operator $d_0 : C^0(X;G) \to C^1(X;G)$ be given by

$$d_0\alpha(u,v) = \alpha(u)\alpha(v)^{-1}.$$

For $\phi \in C^1(X;G)$ and $(u,v,w) \in X_{\mathrm{ord}}(2)$ let

$$d_1\phi(u,v,w) = \phi(u,v)\phi(v,w)\phi(w,u).$$

Note that if $d_1\phi(u_1,u_2,u_3) = 1$, then $d_1\phi(u_{\pi(1)},u_{\pi(2)},u_{\pi(3)}) = 1$ for all permutations π. The set of *G-valued 1-cocycles* of X is given by

$$Z^1(X;G) = \left\{\phi \in C^1(X;G) : d_1\phi(u,v,w) = 1 \text{ for all } (u,v,w) \in X_{\rm ord}(2)\right\}.$$

Define an action of $C^0(X;G)$ on $C^1(X;G)$ as follows. For $\alpha \in C^0(X;G)$ and $\phi \in C^1(X;G)$ let

$$\alpha.\phi(u,v) = \alpha(u)\phi(u,v)\alpha(v)^{-1}.$$

Note that $d_0\alpha = \alpha.1$ and that $Z^1(X;G)$ is invariant under the action of $C^0(X;G)$. For $\phi \in C^1(X;G)$ let $[\phi]$ denote the orbit of ϕ under the action of $C^0(X;G)$. The *first cohomology of X with coefficients in G* is the set of orbits

$$H^1(X;G) = \left\{[\phi] : \phi \in Z^1(X;G)\right\}.$$

Remark 1.2. *(i) If G is abelian, then $H^1(X;G)$ is the usual 1-dimensional cohomology group of X with G coefficients. For a general group G, the first cohomology $H^1(X;G)$ is only a set and need not have a natural group structure.*
(ii) Assume that X is connected and let $\pi_1(X,x_0)$ denote the fundamental group of X with respect to a base point x_0. Define an equivalence relation $\sim$ on the set of homomorphisms $Hom(\pi_1(X,x_0),G)$ by $\varphi_1 \sim \varphi_2$ if there exists a $g \in G$ such that $\phi_2(\gamma) = g\phi_1(\gamma)g^{-1}$ for all $\gamma \in \pi_1(X,x_0)$. Then $H^1(X;G)$ can be identified with $Hom(\pi_1(X,x_0),G)/\sim$. For more details, see Olum's paper [15].

Definition 1.3. *Let X be a simplicial complex and let G be a finite group with a left action on a finite set S. For a 1-cochain $\phi \in C^1(X;G)$, let $Y_\phi = X \times_\phi S$ be the simplicial complex on the vertex set $Y_\phi(0) = \{[u,s] : u \in X(0), s \in S\}$, whose k-simplices are $\tau = \left\{[u_0,s_0],\ldots,[u_k,s_k]\right\}$, where $\{u_0,\ldots,u_k\} \in X(k)$, and $s_i = \phi(u_i,u_j)s_j$ for all $0 \le i,j \le k$. Let $f : Y_\phi \to X$ be the simplicial projection map given by $f([u,s]) = u$.*

Remark 1.4. *(i) If $\alpha \in C^0(X;G)$, then the map $g_\alpha : Y_{\alpha.\phi} \to Y_\phi$ given by $g_\alpha([v,s]) = [v,\alpha(v)^{-1}s]$ is a simplicial isomorphism that satisfies $f = fg_\alpha$, i.e. $Y_{\alpha.\phi} \cong_X Y_\phi$.*
(ii) For $\tilde{u} = [u,s] \in Y_\phi(0)$, the restricted map $f : {\rm st}(Y,\tilde{u}) \to {\rm st}(X,u)$ is an isomorphism iff $\mu_f(\tilde{u}) = 0$. We may thus regard the deficiency $m_f(Y_\phi)$ as a measure of the failure of f to be a covering map.

We next recall the classical connection between G-covering spaces of X and the cohomology set $H^1(X;G)$. See Steenrod [17] for general spaces X, and Surowski [18] for the following simplicial version.

Theorem 1.5 ([18])**.** *Let X be a connected complex. If $\phi \in Z^1(X;G)$ then $f : X \times_\phi S \to X$ is a covering map. Conversely, let $f : Y \to X$ be a simplicial covering map and let $v_0 \in X(0)$. Then there is an action of $G = \pi_1(X,v_0)$ on $S = f^{-1}(v_0)$, and a $\phi \in Z^1(X;G)$ such that $Y \cong_X X \times_\phi S$.*

1.3 Cosystolic 1-Expansion

For $\phi \in C^1(X;G)$ let

$${\rm supp}(\phi) = \left\{\{u,v\} \in X(1) : \phi(u,v) \neq 1\right\}$$

and

$${\rm supp}(d_1\phi) = \left\{\{u,v,w\} \in X(2) : d_1\phi(u,v,w) \neq 1\right\}.$$

The *norms* of ϕ and of $d_1\phi$ are given by

$$\|\phi\| = \sum_{e \in {\rm supp}(\phi)} c_X(e)$$

and

$$\|d_1\phi\| = \sum_{\sigma \in {\rm supp}(d_1\phi)} c_X(\sigma).$$

The *distance* between $\phi, \psi \in C^1(X;G)$ is

$$\operatorname{dist}(\phi,\psi) = \|\phi\psi^{-1}\|.$$

The *cosystolic norm* of $\phi \in C^1(X;G)$ is the distance of ϕ from $Z^1(X;G)$, i.e.

$$\|\phi\|_{\text{csy}} = \min\left\{\|\phi\psi^{-1}\| : \psi \in Z^1(X;G)\right\}.$$

Note that if $\alpha \in C^0(X;G)$, then $\alpha^{-1}.1 \in Z^1(X;G)$ and $\operatorname{supp}\left(\phi(\alpha^{-1}.1)^{-1}\right) = \operatorname{supp}(\alpha.\phi)$. Hence

$$\|\phi\|_{\text{csy}} \leq \|\alpha.\phi\|. \tag{2}$$

The *cosystolic expansion* of $\phi \in C^1(X;G) \setminus Z^1(X;G)$ is

$$h(\phi) = \frac{\|d_1\phi\|}{\|\phi\|_{\text{csy}}}.$$

The *cosystolic expansion* of X is

$$h_1(X;G) = \min\left\{h(\phi) : \phi \in C^1(X;G) \setminus Z^1(X;G)\right\}.$$

Example 1.6 (Proposition 3.1 in [13], Proposition 6.7 in [8])**.** *Let Δ_{n-1} denote the $(n-1)$-simplex. then for any group G*

$$h_1(\Delta_{n-1};G) \geq \frac{n}{n-2}.$$

1.4 Deficiency, Near Coverings and Expansion

Let G be a finite group that acts on a finite set S. For $g \in G$ let $\operatorname{fix}(g) = |\{s \in S : gs = s\}|$. The *fixity* of the action of G on S is $\operatorname{Fix}_G(S) = \max_{g \neq 1} \operatorname{fix}(g)$. The action of G is *faithful* if $\operatorname{Fix}_G(S) < |S|$, and *free* if $\operatorname{Fix}_G(S) = 0$.

Let X be a complex with bounded below expansion $h_1(X;G)$ and let $\phi \in C^1(X;G)$. The following result shows, roughly speaking, that if the deficiency of the projection $X \times_\phi S \to X$ is small, then ϕ is close to a 1-cocycle in $H^1(X;G)$.

Theorem 1.7. *Let G act on a finite set S. Then for any $\phi \in C^1(X;G)$ there exists a $\psi \in Z^1(X;G)$ such that*

$$\operatorname{dist}(\phi,\psi) \leq \frac{m_f(X \times_\phi S)}{\left(1 - \frac{\operatorname{Fix}_G(S)}{|S|}\right) \cdot h_1(X;G)}. \tag{3}$$

Let q be a prime power and let $\mathbb{F}_q$ denote the finite field with q elements. The spherical building $A_n(\mathbb{F}_q)$ is the $(n-1)$-dimensional pure simplicial complex whose vertex set consists of all linear subspaces $0 \neq U \subsetneq \mathbb{F}_q^{n+1}$, with maximal simplices $\{U_1, \ldots, U_n\}$ where $0 \neq U_1 \subsetneq U_2 \subsetneq \cdots \subsetneq U_n \subsetneq \mathbb{F}_q^{n+1}$. Spherical buildings such as $A_n(\mathbb{F}_q)$ play an significant role in a number of fields, including representation theory, topological combinatorics and the emerging field of high dimensional expanders (see, e.g., [12, 16, 3, 10]). The main result of this paper is the following lower bound on the cosystolic expansion of $A_3(\mathbb{F}_q)$.

Theorem 1.8. *For any finite group G*

$$h_1\big(A_3(\mathbb{F}_q);G\big) \geq \frac{1}{9}.$$

Remark 1.9. *(i) The case $G = \mathbb{Z}/2$ of Theorem 1.8 is implicit (with a different constant) in Gromov's paper [7], see also Corollary 3.6 in [11] and Corollary 3.9 in [8]. The general non-abelian case requires somewhat different ideas, and is derived here as a consequence of a lower bound (Theorem 3.1) on the non-abelian* 1*-expansion of order complexes of geometric lattices.*
(ii) Theorem 5.3 in [8] implies that $h_1\big(A_3(\mathbb{F}_q);G\big) \leq 1 + O(q^{-\frac{1}{2}})$. It would be interesting to narrow the gap between this upper bound and the lower bound given in Theorem 1.8.

Combining Theorem 1.7 and Theorem 1.8 we obtain

Corollary 1.10. *Let G act on a finite set S. Then for any $\phi \in C^1(A_3(\mathbb{F}_q);G)$ there exists a $\psi \in Z^1(A_3(\mathbb{F}_q);G)$ such that*

$$\mathrm{dist}(\phi,\psi) \leq \frac{9\, m_f(X \times_\phi S)}{\left(1 - \frac{\mathrm{Fix}_G(S)}{|S|}\right)}.$$

In particular, if the action of G is free then

$$\mathrm{dist}(\phi,\psi) \leq 9\, m_f(X \times_\phi S).$$

Remark 1.11. *The simple connectivity of $A_3(\mathbb{F}_q)$ implies that if $\psi \in Z^1(X;G)$, then Y_ψ is isomorphic to the trivial $|S|$-fold covering of $A_3(\mathbb{F}_q)$.*

The paper is organized as follows. In Section 2 we prove Theorem 1.7 that links between cover-stability and 1-expansion. In Section 3 we study the expansion of geometric lattices (Theorem 3.1), and then use a symmetry argument to deduce a lower bound on $h_1(A_3(\mathbb{F}_q);G)$ (Theorem 1.8) . We conclude in Section 4 with some comments and questions.

2 Cosystolic Expansion and Cover Stability

Proof of Theorem 1.7. Let $\phi \in C^1(X;G) \setminus Z^1(X;G)$. Recall that $f : Y_\phi = X \times_\phi S \to X$ is the projection map $f([u,s]) = u$. Let $[u,s] \in Y_\phi(0)$ and let $e = \{v_1,v_2\} \in \mathrm{lk}(X,u)$. Then $e \in f\left(\mathrm{lk}(Y_\phi,[u,s])\right)$ iff there exist $s_1,s_2 \in S$ such that $\{[u,s],[v_1,s_1],[v_2,s_2]\} \in Y_\phi$, i.e. iff $s_1 = \phi(v_1,u)s$, $s_2 = \phi(v_2,u)s$, and $s_1 = \phi(v_1,v_2)s_2$. Writing the last equality in terms of first two, we obtain

$$\phi(v_1,u)s = s_1 = \phi(v_1,v_2)s_2 = \phi(v_1,v_2)\phi(v_2,u)s,$$

and therefore

$$s = \phi(u,v_1)\phi(v_1,v_2)\phi(v_2,u)s = d_1\phi(u,v_1,v_2)s.$$

It follows that

$$e = \{v_1,v_2\} \in D_f([u,s]) \quad \Longleftrightarrow \quad d_1\phi(u,v_1,v_2)s \neq s. \tag{4}$$

Using Eqs. (4) and (1) respectively for steps (a) and (b) below, we compute

$$\begin{aligned}
|S|\, m_f(X \times_\phi S) &= \sum_{u\in X(0)} c_X(u) \sum_{\tilde{u}\in f^{-1}(u)} \mu_f(\tilde{u}) \\
&= \sum_{u\in X(0)} c_X(u) \sum_{s\in S} \mu_f([u,s]) \\
&= \sum_{u\in X(0)} c_X(u) \sum_{s\in S} \sum_{e\in D_f([u,s])} c_{\mathrm{lk}(X,u)}(e) \\
&\overset{(a)}{=} \sum_{u\in X(0)} c_X(u) \sum_{\{v_1,v_2\}\in \mathrm{lk}(X,u)(1)} |\{s : d_1\phi(u,v_1,v_2)s \neq s\}| \cdot c_{\mathrm{lk}(X,u)}(\{v_1,v_2\}) \\
&= \sum_{u\in X(0)} c_X(u) \sum_{e\in \mathrm{lk}(X,u)(1)} \left(|S| - \mathrm{fix}\left(d_1\phi(u\cup e)\right)\right) \cdot c_{\mathrm{lk}(X,u)}(e) \\
&\geq \left(|S| - \mathrm{Fix}_G(S)\right) \sum_{u\in X(0)} \sum_{\{e\in \mathrm{lk}(X,u)(1) : d_1\phi(u\cup e)\neq 1\}} c_X(u) c_{\mathrm{lk}(X,u)}(e) \\
&\overset{(b)}{=} \left(|S| - \mathrm{Fix}_G(S)\right) \sum_{u\in X(0)} \sum_{\{e\in \mathrm{lk}(X,u)(1) : u\cup e \in \mathrm{supp}(d_1\phi)\}} \frac{1}{3} \cdot c_X(u\cup e) \\
&= \left(|S| - \mathrm{Fix}_G(S)\right) \|d_1\phi\|.
\end{aligned}$$

As $h_1(X;G) \le \frac{\|d_1\phi\|}{\|\phi\|_{\mathrm{csy}}}$, it follows that

$$\begin{aligned}\min\big\{\mathrm{dist}(\phi,\psi) : \psi \in Z^1(X;G)\big\} &= \|\phi\|_{\mathrm{csy}} \\ &\le \frac{\|d_1\phi\|}{h_1(X;G)} \le \frac{|S|\cdot m_f(X\times_\phi S)}{\big(|S| - \mathrm{Fix}_G(S)\big)\cdot h_1(X;G)} \\ &= \frac{m_f(X\times_\phi S)}{\big(1-\frac{\mathrm{Fix}_G(S)}{|S|}\big)\cdot h_1(X;G)} \quad .\end{aligned}$$

□

3 The 1-Expansion of Geometric Lattices

Let $(P,\le)$ be a finite poset. The *order complex* of P is the simplicial complex on the vertex set P whose simplices are the chains $a_0 < \cdots < a_k$ of P. In the sequel we identify a poset with its order complex. A poset $(L,\le)$ is a *lattice* if any two elements $x, y \in L$ have a unique minimal upper bound $x \vee y$ and a unique maximal lower bound $x \wedge y$. A lattice L with minimal element $\widehat{0}$ and maximal element $\widehat{1}$ is *ranked*, with rank function $\mathrm{rank}(\cdot)$, if $\mathrm{rank}(\widehat{0}) = 0$ and $\mathrm{rank}(y) = \mathrm{rank}(x) + 1$ whenever y covers x (i.e., y is a minimal element in $\{z : z > x\}$). L is a *geometric lattice* if $\mathrm{rank}(x)+\mathrm{rank}(y) \ge \mathrm{rank}(x\vee y)+\mathrm{rank}(x\wedge y)$ for any $x, y \in L$, and any element in L is a join of atoms (i.e., rank 1 elements).

Let L be a geometric lattice with $\mathrm{rank}(\widehat{1}) = n \ge 3$. A classical result of Folkman [5] asserts that $\overline{L} = L - \{\widehat{0},\widehat{1}\}$ is homotopy equivalent to a wedge of $(n-2)$-spheres. In particular, $\overline{L}$ is simply connected, and hence $H^1(\overline{L};G) = \{1\}$ for any group G. Here we provide a lower bound for $h_1(\overline{L};G)$. Let S be a set of linear orderings on the set of atoms A of L, equipped with a probability distribution μ. Let $\prec_s$ denote the ordering associated with $s \in S$. For $s \in S$ and $v \in L \setminus \{\widehat{0}\}$ let $b(s,v) = \min\{a \in A : a \le v\}$ where the minimum is taken with respect to $\prec_s$. Note that $b(s,\widehat{1})$ is the $\prec_s$-minimal element of A. For $s \in S$ and $v_0 < v_1 \in \overline{L}$, let $a_0 = b(s,v_0), a_1 = b(s,v_1), a_2 = b(s,\widehat{1})$. Clearly $a_2 \preceq_s a_1 \preceq_s a_0$. Let $Y_s(v_0v_1)$ be the 2-dimensional subcomplex of $\overline{L}$ depicted in Figure 1. For a fixed $\tau \in \overline{L}(2)$, let $\delta_s(\tau)$ be the random variable on S given by

$$\delta_s(\tau) := \sum_{\{v_0v_1 \in \overline{L}(1) : \tau \in Y_s(v_0v_1)\}} \frac{c_X(v_0v_1)}{c_X(\tau)}, \tag{5}$$

and let $\delta(\tau) = E[\delta_s(\tau)] = \sum_{s\in S}\mu(s)\delta_s(\tau)$ denote its expectation. The next result may be viewed as a non-commutative homotopical counterpart of the 2-dimensional case of Theorem 2.5 in [8].

Theorem 3.1.

$$h_1(\overline{L};G) \ge \left(\max_{\tau\in\overline{L}(2)} \delta(\tau)\right)^{-1} .$$

We will need the following simple fact.

Claim 3.2. *Let G be a group and let K be a 2-dimensional simplicial complex such that $H^1(K;G) = \{1\}$. Suppose $x_0, \ldots, x_{m-1}, x_m = x_0$ are the vertices of a 1-cycle in K. If $\phi \in C^1(K;G)$ satisfies*

$$\phi(x_0,x_1)\cdot\phi(x_1,x_2)\cdots\phi(x_{m-1},x_0) \ne 1 \tag{6}$$

then there exists a 2-simplex $(a,b,c) \in K_{ord}(2)$ such that $d_1\phi(a,b,c) \ne 1$.

Proof. Suppose to the contrary that $d_1\phi = 1$. As $H^1(K;G) = \{1\}$ it follows that $\phi = d_0\alpha$ for some $\alpha \in C^0(K;G)$. Hence

$$\phi(x_0,x_1)\cdot\phi(x_1,x_2)\cdots\phi(x_{m-1},x_0) = \big(\alpha(x_0)\alpha(x_1)^{-1}\big)\cdot\big(\alpha(x_1)\alpha(x_2)^{-1}\big)\cdots\big(\alpha(x_{m-1})\alpha(x_0)^{-1}\big) = 1,$$

in contradiction with (6).

□

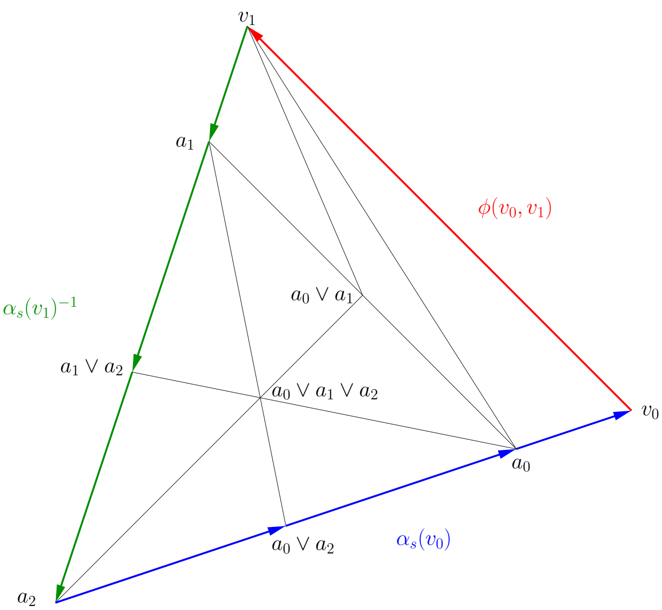

Figure 1: The subcomplex $Y_s(v_0v_1)$ and the cochain $(\alpha_s).\phi(v_0, v_1) = \alpha_s(v_0)\phi(v_0, v_1)\alpha_s(v_1)^{-1}$.

Proof of Theorem 3.1: Let $\phi \in C^1(\overline{L}; G)$. For $s \in S$, define $\alpha_s \in C^0(\overline{L}; G)$ by

$$\alpha_s(v) = \phi\big(b(s, \widehat{1}), b(s, \widehat{1}) \vee b(s, v)\big) \cdot \phi\big(b(s, \widehat{1}) \vee b(s, v), b(s, v)\big) \cdot \phi\big(b(s, v), v\big).$$

Let $v_0 < v_1 \in \overline{L}$ and, as before, denote $a_0 = b(s, v_0), a_1 = b(s, v_1), a_2 = b(s, \widehat{1})$. Consider the 1-cycle in $Y_s(uv)$ whose vertices are

$$(x_0, \ldots, x_7) = (a_2, a_0 \vee a_2, a_0, v_0, v_1, a_1, a_1 \vee a_2, a_2).$$

Then (see Figure 1):

$$\begin{aligned}(\alpha_s).\phi(v_0, v_1) &= \alpha_s(v_0)\phi(v_0, v_1)\alpha_s(v_1)^{-1} \\ &= \phi(x_0, x_1)\phi(x_1, x_2) \cdots \phi(x_5, x_6)\phi(x_6, x_0).\end{aligned} \tag{7}$$

Since $Y_s(v_0v_1)$ is contractible, it follows from (7) and Claim 3.2 that

$$\big\{v_0v_1 \in \overline{L}(1) : (\alpha_s).\phi(v_0, v_1) \neq 1\big\} \subset \big\{v_0v_1 \in \overline{L}(1) : \mathrm{supp}(d_1\phi) \cap Y_s(v_0v_1) \neq \emptyset\big\}. \tag{8}$$

Using (2) and (8) respectively for steps (a) and (b) below, we obtain

$$\begin{aligned}\|\phi\|_{\mathrm{csy}} &\overset{(a)}{\leq} \sum_{s \in S} \mu(s)\|(\alpha_s).\phi\| \\ &= \sum_{s \in S} \mu(s) \sum \{c_X(v_0v_1) : v_0v_1 \in \overline{L}(1), (\alpha_s).\phi(v_0, v_1) \neq 1\} \\ &\overset{(b)}{\leq} \sum_{s \in S} \mu(s) \sum \{c_X(v_0v_1) : v_0v_1 \in \overline{L}(1), \mathrm{supp}(d_1\phi) \cap Y_s(v_0v_1) \neq \emptyset\} \\ &\leq \sum_{s \in S} \mu(s) \sum_{\tau \in \mathrm{supp}(d_1\phi)} \sum \{c_X(v_0v_1) : v_0v_1 \in \overline{L}(1), \tau \in Y_s(v_0v_1)\} \\ &= \sum_{\tau \in \mathrm{supp}(d_1\phi)} c_X(\tau) \sum_{s \in S} \mu(s) \sum \left\{\frac{c_X(v_0v_1)}{c_X(\tau)} : v_0v_1 \in \overline{L}(1), \tau \in Y_s(v_0v_1)\right\} \\ &= \sum_{\tau \in \mathrm{supp}(d_1\phi)} c_X(\tau) E[\delta_s(\tau)] \\ &= \sum_{\tau \in \mathrm{supp}(d_1\phi)} c_X(\tau)\delta(\tau) \leq \|d_1\phi\| \max_{\tau \in \overline{L}(2)} \delta(\tau).\end{aligned} \tag{9}$$

□

For lattices L with sufficient symmetry (e.g. spherical buildings), Theorem 3.1 can be used to give explicit lower bounds on $h_1(\overline{L}, G)$.

Proof of Theorem 1.8: Let L be the lattice of all linear subspaces of $\mathbb{F}_q^4$ ordered by inclusion. Then $\overline{L} = A_3(\mathbb{F}_q)$. Let $\prec$ be an arbitrary fixed linear order on the set of atoms A. Let S be the group $GL_4(\mathbb{F}_q)$ with the natural action on $\overline{L}$. Equip S with the uniform distribution. For $s \in S$ let $\prec_s$ be the linear order on A given by $a \prec_s a'$ if $s^{-1}a \prec s^{-1}a'$. Let id denote the identity element of S.

Claim 3.3. *Let $s, t \in S$, $e \in \overline{L}(1)$ and $\tau \in \overline{L}(2)$. Then:*
(i) $b(s, sv) = sb(\mathrm{id}, v)$ for any $v \in L \setminus \{\widehat{0}\}$.
(ii) $Y_s(se) = sY_{\mathrm{id}}(e)$.
(iii) $tY_s(e) = Y_{ts}(te)$.

Proof. (i) First note that $b(\mathrm{id}, v) \leq v$ and therefore $sb(\mathrm{id}, v) \leq sv$. Moreover, if $y \in A$ satisfies $y \leq sv$, then $s^{-1}y \leq v$. Hence $s^{-1}(sb(\mathrm{id}, v)) = b(\mathrm{id}, v) \preccurlyeq s^{-1}y$ and so $sb(\mathrm{id}, v) \preccurlyeq_s y$. It follows that $sb(\mathrm{id}, v) = b(s, sv)$.
(ii) Let $e = v_0v_1$. Writing $v_2 = \widehat{1}$, it follows by (i) that the vertex set of $Y_s(se)$ is

$$\begin{aligned} Y_s(se)(0) &= \{sv_0, sv_1\} \cup \left\{ \bigvee_{i \in I} b(s, sv_i) : \emptyset \neq I \subset \{0,1,2\} \right\} \\ &= \{sv_0, sv_1\} \cup \left\{ \bigvee_{i \in I} sb(\mathrm{id}, v_i) : \emptyset \neq I \subset \{0,1,2\} \right\} \\ &= sY_{\mathrm{id}}(e)(0). \end{aligned}$$

As the action of S preserves incidences, it follows that $Y_s(se) = sY_{\mathrm{id}}(e)$.
(iii) Using (ii) in equalities (a) and (b) below, we obtain

$$\begin{aligned} tY_s(e) = tY_s\left(s(s^{-1}e)\right) &\overset{(a)}{=} tsY_{\mathrm{id}}(s^{-1}e) \\ &\overset{(b)}{=} Y_{ts}\left((ts)s^{-1}e\right) = Y_{ts}(te). \end{aligned}$$

□

For $\tau \in \overline{L}(2)$ let

$$R(\tau) = \{(s, e) \in S \times \overline{L}(1) : \tau \in Y_s(e)\}.$$

As $\frac{c_{\overline{L}}(e)}{c_{\overline{L}}(\tau)} = \frac{q+1}{3}$, it follows that

$$\delta(\tau) = \frac{1}{|S|} \sum_{s \in S} \delta_s(\tau) = \frac{1}{|S|} \sum_{s \in S} \sum_{\{e \in \overline{L}(1) : \tau \in Y_s(e)\}} \frac{c_X(e)}{c_X(\tau)} = \frac{q+1}{3|S|} |R(\tau)|. \tag{10}$$

Claim 3.3(iii) implies that for any $t \in S$ the map $(s, e) \to (ts, te)$ is a bijection from $R(\tau)$ to $R(t\tau)$. Together with (10), it follows that $\delta(\tau) = \delta(t\tau)$. Next note that S is transitive on $\overline{L}(2)$ and thus δ is constant, i.e. $\delta(\tau) = \gamma$ for all $\tau \in \overline{L}(2)$. Therefore

$$\begin{aligned} f_2(\overline{L})\gamma &= \sum_{\tau \in \overline{L}(2)} \delta(\tau) = \sum_{\tau \in \overline{L}(2)} \frac{1}{|S|} \sum_{s \in S} \delta_s(\tau) \\ &= \frac{q+1}{3|S|} |\{(s, e, \tau) \in S \times \overline{L}(1) \times \overline{L}(2) : \tau \in Y_s(e)\}| \\ &= \frac{q+1}{3|S|} \sum_{s \in S} \sum_{e \in \overline{L}(1)} f_2(Y_s(e)) \\ &\leq \frac{q+1}{3|S|} \cdot |S| \cdot f_1(\overline{L}) \cdot 9 = 3(q+1) f_1(\overline{L}). \end{aligned}$$

Therefore

$$\gamma \leq \frac{3(q+1)f_1(\overline{L})}{f_2(\overline{L})} = 9,$$

hence Theorem 1.8 follows from Theorem 3.1.

$\square$

4 Concluding Remarks

In this paper we showed that if X has a bounded below 1-expansion over a finite group G, then X is cover-stable in the sense that if the projection $Y_\phi \to X$ has small deficiency, then ϕ is close to a 1-cocycle. Together with a lower bound on $h_1(A_3(\mathbb{F}_q);G)$, this implied that $A_3(\mathbb{F}_q)$ is cover-stable. Our work suggests some natural questions regarding topological or algebraic situations where only partial information concerning local structure is available.

- Most lower bounds on cosystolic expansion obtained so far (see e.g. [14, 7, 4, 11, 8]) depend on an averaging and symmetry technique that seems applicable only in fairly restricted situations. It would be very useful to devise additional methods that could handle more general families of complexes, e.g. bounded degree complexes.

- Our results are concerned with cosystolic 1-expansion with G coefficients. A related, but inequivalent, notion of expansion that plays a key role in local to global results is spectral expansion. Originated with Garland's seminal work on the real cohomology of p-adic groups [6], spectral expansion has found numerous applications in areas ranging from groups with property (T) [2] and hypergraph matching theory [1], to theoretical computer science [10]. It would be interesting to establish cover-stability results based on spectral expansion. One reason is that in contrast with its cosystolic/coboundary counterparts, spectral expansion can often be estimated efficiently.

- It would be interesting to formulate stability versions of other, non-discrete, topological classification theorems. For example, as mentioned earlier, complex line bundles over compact X are parametrized by $H^2(X;\mathbb{Z})$. Can one introduce a notion of cosystolic expansion on X, together with a continuous version of the deficiency of a map and an appropriate measure of proximity of integral 2-cochains, that would lead to an analogue of Theorem 1.7 for complex line bundles?

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