Maximal Generalized Rank in Graphical Matrix Spaces

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Abstract

Let $M_n(\mathbb{F})$ be the space of $n \times n$ matrices over a field $\mathbb{F}$. For a subset $\mathcal{B} \subset [n]^2$ let $M_{\mathcal{B}}(\mathbb{F}) = \{A \in M_n(\mathbb{F}) : A(i,j) = 0 \text{ for } (i,j) \notin \mathcal{B}\}$. Let $\nu_b(\mathcal{B})$ denote the matching number of the n by n bipartite graph determined by $\mathcal{B}$. For $S \subset M_n(\mathbb{F})$ let $\rho(S) = \max\{\text{rk}(A) : A \in S\}$. Li, Qiao, Wigderson, Wigderson and Zhang (arXiv:2206.04815, 2022) have recently proved the following characterization of the maximal dimension of bounded rank subspaces of $M_{\mathcal{B}}(\mathbb{F})$.

Theorem (Li, Qiao, Wigderson, Wigderson, Zhang)**.** *For any $\mathcal{B} \subset [n]^2$*

$$\max\{\dim W : \ W \leq M_{\mathcal{B}}(\mathbb{F}) \ , \ \rho(W) \leq k\} = \max\{|\mathcal{B}'| : \ \mathcal{B}' \subset \mathcal{B} \ , \ \nu_b(\mathcal{B}') \leq k\} . \tag{1}$$

The main results of this note are two extensions of (1). Let $\mathbb{S}_n$ denote the symmetric group on $[n]$. For $\omega : \coprod_{n=1}^{\infty} \mathbb{S}_n \to \mathbb{F}^* = \mathbb{F} \setminus \{0\}$ define a function D_ω on each $M_n(\mathbb{F})$ by $D_\omega(A) = \sum_{\sigma \in \mathbb{S}_n} \omega(\sigma) \prod_{i=1}^n A(i, \sigma(i))$. Let $\text{rk}_\omega(A)$ be the maximal k such that there exists a $k \times k$ submatrix B of A with $D_\omega(B) \neq 0$. For $S \subset M_n(\mathbb{F})$ let $\rho_\omega(S) = \max\{\text{rk}_\omega(A) : A \in S\}$. The first extension of (1) concerns general weight functions.

Theorem. *For any ω as above and $\mathcal{B} \subset [n]^2$*

$$\max\{\dim W : \ W \leq M_{\mathcal{B}}(\mathbb{F}) \ , \ \rho_\omega(W) \leq k\} = \max\{|\mathcal{B}'| : \mathcal{B}' \subset \mathcal{B} \ , \ \nu_b(\mathcal{B}') \leq k\} .$$

Let $A_n(\mathbb{F})$ denote the space of alternating matrices in $M_n(\mathbb{F})$. For a graph $\mathcal{G} \subset \binom{[n]}{2}$ let $A_{\mathcal{G}}(\mathbb{F}) = \{A \in A_n(\mathbb{F}) : A(i,j) = 0 \text{ if } \{i,j\} \notin \mathcal{G}\}$. Let $\nu(\mathcal{G})$ denote the matching number of $\mathcal{G}$. The second extension of (1) concerns general graphs.

Theorem. *For any $\mathcal{G} \subset \binom{[n]}{2}$*

$$\max\{\dim U : \ U \leq A_{\mathcal{G}}(\mathbb{F}) \ , \ \rho(U) \leq 2k\} = \max\{|\mathcal{G}'| : \ \mathcal{G}' \subset \mathcal{G} \ , \ \nu(\mathcal{G}') \leq k\} .$$

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1 Introduction

Let $M_n(\mathbb{F})$ denote the space of $n \times n$ matrices over a field $\mathbb{F}$. For $A \in M_n(\mathbb{F})$ and subsets $\emptyset \neq I = \{i_1, \ldots, i_k\}, J = \{j_1, \ldots, j_k\} \subset [n] := \{1, \ldots, n\}$ such that $i_1 < \cdots < i_k$ and $j_1 < \cdots < j_k$ let $B = A[I|J] \in M_k(\mathbb{F})$ be given by $B(\alpha, \beta) = A(i_\alpha, j_\beta)$ for $1 \leq \alpha, \beta \leq k$. For a vector space W we write $V \leq W$ if V is a linear subspace of W. Let $\binom{[n]}{k}$ denote the family of all k-element subsets of $[n]$. For column vectors $u_1, u_2 \in \mathbb{F}^n$ let $u_1 \otimes u_2 = u_1 \cdot u_2^t \in M_n(\mathbb{F})$. The tensor product of two linear subspaces $U_1, U_2 \leq \mathbb{F}^n$ is given by $U_1 \otimes U_2 = \text{span}\{u_1 \otimes u_2 : u_1 \in U_1, u_2 \in U_2\}$. For a subset $S \subset M_n(\mathbb{F})$ let $\rho(S) = \max\{\text{rk}(A) : A \in S\}$ denote the maximal rank of a matrix in S. The following result was proved by Flanders [4] under the assumption $|\mathbb{F}| \geq k + 1$, and in [8] for all fields.

Theorem 1.1 ([4, 8]). *Let $W \leq M_n(\mathbb{F})$ be a linear subspace such that $\rho(W) \leq k$. Then: (i) $\dim W \leq kn$. (ii) $\dim W = kn$ iff $W = U \otimes \mathbb{F}^n$ or $W = \mathbb{F}^n \otimes U$ for some k-dimensional linear subspace $U \leq \mathbb{F}^n$.*

For $i \in [n]$ let e_i denote the i-th unit vector in $\mathbb{F}^n$. For a subset $\mathcal{B} \subset [n]^2$ let

$$M_{\mathcal{B}}(\mathbb{F}) = \text{span}\{e_i \otimes e_j : (i, j) \in \mathcal{B}\}.$$

A *bipartite matching* in $\mathcal{B}$ is a subset $\mathcal{B}_0 \subset \mathcal{B}$ such that if $(i, j) \neq (i', j') \in \mathcal{B}_0$ then $i \neq i'$ and $j \neq j'$. The *bipartite matching number* of $\mathcal{B}$ is

$$\nu_b(\mathcal{B}) := \max\{|\mathcal{B}_0| : \mathcal{B}_0 \text{ is a bipartite matching in } \mathcal{B}\}.$$

Li, Qiao, Wigderson, Wigderson and Zhang [6] have recently established the following

Theorem 1.2 ([6]). *For any $\mathcal{B} \subset [n]^2$*

$$\max\{\dim W : W \leq M_{\mathcal{B}}(\mathbb{F}), \ \rho(W) \leq k\} = \max\left\{|\mathcal{B}'| : \mathcal{B}' \subset \mathcal{B}, \ \nu_b(\mathcal{B}') \leq k\right\}. \quad (2)$$

Remark 1.3. *When $\mathcal{B} = [n]^2$ Theorem 1.2 specializes to Theorem 1.1(i). Indeed, Kőnig's Theorem (see e.g. Theorem 3.1.11 in [10]) implies that* $\max\left\{|\mathcal{B}'| : \mathcal{B}' \subset [n]^2, \nu_b(\mathcal{B}') \leq k\right\} = kn$.

In this paper we give some extensions of Theorems 1.1 and 1.2. Let $\mathbb{S}_n$ denote the symmetric group on $[n]$. For a weight function $\omega : \coprod_{n=1}^{\infty} \mathbb{S}_n \to \mathbb{F}^* = \mathbb{F} \setminus \{0\}$ let $D_\omega : \coprod_{n=1}^{\infty} M_n(\mathbb{F}) \to \mathbb{F}$ be defined on $A = (A(i, j))_{i,j=1}^n \in M_n(\mathbb{F})$ by

$$D_\omega(A) = \sum_{\sigma \in \mathbb{S}_n} \omega(\sigma) \prod_{i=1}^{n} A(i, \sigma(i)).$$

Remark 1.4. *The functions D_ω were considered by de Seguins Pazzis under the name of* Schur matrix functionals. *See his paper [2] for an in-depth study of the linear preservers of various D_ω's.*

The *ω-rank* $\text{rk}_\omega(A)$ of a matrix $A \in M_n(\mathbb{F})$ is the maximal k such that there exist $I, J \in \binom{[n]}{k}$ such that $D_\omega(A[I|J]) \neq 0$. For $S \subset M_n(\mathbb{F})$ let $\rho_\omega(S) = \max\{\text{rk}_\omega(A) : A \in S\}$. Note that for the sign function $D_{\text{sgn}}(A) = \det A$ and $\text{rk}_{\text{sgn}}(A) = \text{rk}(A)$. Let $\mathbb{1}$ be the constant function $\mathbb{1}(\sigma) \equiv 1$. Then $D_{\mathbb{1}}(A) = \text{per}A$ is the permanent of A, and $\text{prk}(A) := \text{rk}_{\mathbb{1}}(A)$ is the *permanental rank* of A. See [5] for applications of permanental rank to linear preserver problems. Our first result is an extension of Theorem 1.2 to general weight functions.

Theorem 1.5. *For any $\omega : \coprod_{n=1}^{\infty} \mathbb{S}_n \to \mathbb{F}^*$ and any $\mathcal{B} \subset [n]^2$*

$$\max\{\dim W : \ W \le M_{\mathcal{B}}(\mathbb{F}) \ , \ \rho_\omega(W) \le k\} = \max\left\{|\mathcal{B}'| : \mathcal{B}' \subset \mathcal{B} \ , \ \nu_b(\mathcal{B}') \le k\right\}. \qquad (3)$$

Remark 1.6. *As with Theorems 1.1 and 1.2, Theorem 1.5 holds for spaces of rectangular matrices as well. This can be shown e.g. by embedding a subspace of $m \times n$ matrices with $m \le n$ into $M_n(\mathbb{F})$ and then using the present square version of the theorem.*

Specializing Theorem 1.5 to $\mathcal{B} = [n]^2$ and $\omega = \mathbb{1}$ we obtain the following permanental counterpart of Theorem 1.1(i).

Corollary 1.7. *Let $W \le M_n(\mathbb{F})$ be a linear subspace such that $\mathrm{prk}(A) \le k$ for all $A \in W$. Then $\dim W \le kn$.*

For the fields of characteristic 2, we have det = per and thus the equality cases in Corollary 1.7 are those given in Theorem 1.1(ii). For $(k, n) = (1, 2)$ it can be checked that the only 2-dimensional subspaces $W \le M_2(\mathbb{F})$ such that $\dim W = 2$ and $\rho_{\mathbb{1}}(W) = 1$ are W_u and its transpose, where for $0 \ne u = (a, b) \in \mathbb{F}^2$

$$W_u = \left\{ \begin{pmatrix} ax & ay \\ -bx & by \end{pmatrix} : x, y \in \mathbb{F} \right\}.$$

In general we have the following

Theorem 1.8. *Suppose that $\mathrm{char}\,\mathbb{F} \ne 2$, $k \le n$ and $n \ge 3$. Then $W \le M_n(\mathbb{F})$ satisfies $\rho_{\mathbb{1}}(W) = k$ and $\dim W = kn$ iff $W = \mathrm{span}\{e_i\}_{i\in I} \otimes \mathbb{F}^n$ or $W = \mathbb{F}^n \otimes \mathrm{span}\{e_i\}_{i \in I}$ for some $I \in \binom{[n]}{k}$.*

A matrix $A = \left(A(i,j)\right)_{i,j=1}^n \in M_n(\mathbb{F})$ is *alternating* if $A = -A^t$ and $A(i,i) = 0$ for $1 \le i \le n$. Let $A_n(\mathbb{F})$ denote the space of alternating matrices in $M_n(\mathbb{F})$. Recall that $\mathrm{rk}(A)$ is even for all $A \in A_n(\mathbb{F})$, and that $\mathrm{rk}(A) = 2k$ iff there exists a subset $I \in \binom{[n]}{2k}$ such that the principal submatrix $A[I|I]$ is nonsingular. For $u, v \in \mathbb{F}^n$ let $u \wedge v = u \otimes v - v \otimes u \in A_n(\mathbb{F})$. For a subset $\mathcal{G} \subset \binom{[n]}{2}$ let

$$A_{\mathcal{G}}(\mathbb{F}) = \mathrm{span}\{e_i \wedge e_j : \{i,j\} \in \mathcal{G}\}.$$

A *matching* in $\mathcal{G}$ is a subset $\mathcal{G}_0 \subset \mathcal{G}$ such that $f \cap f' = \emptyset$ for all $f \ne f' \in \mathcal{G}_0$. The *matching number* of $\mathcal{G}$ is

$$\nu(\mathcal{G}) := \max\{|\mathcal{G}_0| : \mathcal{G}_0 \text{ is a matching in } \mathcal{G}\}.$$

Our final result is an extension of Theorem 1.2 to spaces of alternating matrices supported on general graphs.

Theorem 1.9. *For any $\mathcal{G} \subset K_n := \binom{[n]}{2}$*

$$\max\{\dim U : \ U \le A_{\mathcal{G}}(\mathbb{F}) \ , \ \rho(U) \le 2k\} = \max\left\{|\mathcal{G}'| : \ \mathcal{G}' \subset \mathcal{G} \ , \ \nu(\mathcal{G}') \le k\right\}. \qquad (4)$$

Remark 1.10. *Theorem 1.9 implies Theorem 1.2. Indeed, given $\mathcal{B} \subset [n]^2$ let $\mathcal{G}$ denote the bipartite graph with sides $\{1, \ldots, n\}$ and $\{n+1, \ldots, 2n\}$ corresponding to $\mathcal{B}$, i.e.*

$$\mathcal{G} = \{\{i, j+n\} : (i,j) \in \mathcal{B}\} \subset \binom{[2n]}{2}.$$

It is straightforward to check that

$$\max\{\dim W:\ W\leq M_{\mathcal{B}}(\mathbb{F})\ ,\ \rho(W)\leq k\}=\max\{\dim U:\ U\leq A_{\mathcal{G}}(\mathbb{F})\ ,\ \rho(U)\leq 2k\}$$

and

$$\max\left\{|\mathcal{B}'|:\ \mathcal{B}'\subset\mathcal{B}\ ,\ \nu_b(\mathcal{B}')\leq k\right\}=\max\left\{|\mathcal{G}'|:\ \mathcal{G}'\subset\mathcal{G}\ ,\ \nu(\mathcal{G}')\leq k\right\}.$$

Thus (2) follows from (4).

The paper is organized as follows. In Section 2 we prove Theorem 1.5. Our main tool is a combinatorial lower bound on the maximal ω-rank in a subspace of matrices given in Proposition 2.2. In Section 3 we prove Theorem 1.8. In Section 4 we use a result from [9] to establish Theorem 1.9. We conclude in Section 5 with some remarks and open problems.

2 Maximal ω-Rank in Subspaces of $M_{\mathcal{B}}(\mathbb{F})$

In this section we prove Theorem 1.5. We will need the following facts.

Claim 2.1. *For any $\mathcal{B}\subset[n]^2$*

$$\rho_\omega(M_{\mathcal{B}}(\mathbb{F}))=\nu_b(\mathcal{B}). \tag{5}$$

Proof. Let $\rho_\omega(M_{\mathcal{B}}(\mathbb{F}))=k$ and $\nu_b(\mathcal{B})=\ell$. Then there exists a matrix $A\in M_{\mathcal{B}}(\mathbb{F})$ and a $k\times k$ submatrix $A'=A[I|J]$ such that $D_\omega(A')\neq 0$. Let $I=\{i_1<\cdots<i_k\}$ and $J=\{j_1<\cdots<j_k\}$. Then

$$0\neq D_\omega(A')=\sum_{\pi\in\mathbb{S}_k}\omega(\pi)\prod_{t=1}^{k}A(i_t,j_{\pi(t)}).$$

It follows that there exists $\pi\in\mathbb{S}_k$ such that $A(i_t,j_{\pi(t)})\neq 0$ for all $1\leq t\leq k$. Thus $\mathcal{B}_0=\left\{(i_t,j_{\pi(t)})\right\}_{t=1}^{k}$ is a bipartite matching of size k in $\mathcal{B}$ and therefore $\ell\geq k$. For the other direction, let $\mathcal{B}_0=\{(i_t,j_t)\}_{t=1}^{\ell}$ be a bipartite matching of size ℓ in $\mathcal{B}$. By reordering we may assume that $i_1<\cdots<i_\ell$. Let $\pi\in\mathbb{S}_\ell$ be such that $j_{\pi(1)}<\cdots<j_{\pi(\ell)}$. Let $I=\{i_1,\ldots,i_\ell\}$, $J=\{j_1,\ldots,j_\ell\}$, and let $A=\sum_{t=1}^{\ell}e_{i_t}\otimes e_{j_t}\in M_{\mathcal{B}}(\mathbb{F})$. Then $A'=A[I|J]$ satisfies $D_\omega(A')=\omega(\pi^{-1})\neq 0$ and therefore $k\geq \mathrm{rk}_\omega(A)=\ell$.

□

Let $\prec$ be the lexicographic order on $[n]^2$, i.e. $(i,j)\prec(i',j')$ if either $i<i'$ or $i=i'$ and $j<j'$. For $0\neq A\in M_n(\mathbb{F})$ let $q(A)=\min_\prec\{(i,j):A(i,j)\neq 0\}$. For $S\subset M_n(\mathbb{F})$ let $\mathbb{B}(S)=\{q(A):A\in S\}$. Note that if $W\leq M_n(\mathbb{F})$ is a d-dimensional linear subspace then $|\mathbb{B}(W)|=d=\dim W$. Indeed, performing Gaussian elimination on a arbitrary basis of W according the order $\prec$, we obtain another basis $\{A_1,\ldots,A_d\}$ such that $q(A_1),\ldots,q(A_d)$ are distinct and then $\mathbb{B}(W)=\{q(A_1),\ldots,q(A_d)\}$. The main tool in the proof of Theorem 1.5 is the following

Proposition 2.2. *Let $W\leq M_n(\mathbb{F})$ be a linear subspace. Then*

$$\rho_\omega(W)\geq\nu_b(\mathbb{B}(W)). \tag{6}$$

Remark 2.3. *The case $\omega = sgn$ of Proposition 2.2 is equivalent to Theorem 1 in [8]. The proof for general weight functions given below requires an additional idea.*

Proof of Proposition 2.2. Let $k = \nu_b(\mathbb{B}(W))$. Then there exist $A_1, \ldots, A_k \in W$ with $q(A_t) = (i_t, j_t)$ such that $\{(i_t, j_t)\}_{t=1}^k$ is a bipartite matching of size k. Let $I = \{i_1, \ldots, i_k\}, J = \{j_1, \ldots, j_k\}$. By reordering and rescaling the matrices A_t's, we may assume that $i_1 < \cdots < i_k$ and that $A_t(i_t, j_t) = 1$. Let $\pi \in \mathbb{S}_k$ be such that $j_{\pi(1)} < \cdots < j_{\pi(k)}$. For $1 \leq t \leq k$ let $C_t = A_t[I|J] \in M_k(\mathbb{F})$. Note that

$$C_t(\alpha, \beta) = 0 \text{ if } \alpha < t, \tag{7}$$

$$C_t(t, \beta) = 0 \text{ if } \beta < \pi^{-1}(t), \tag{8}$$

and

$$C_t(t, \pi^{-1}(t)) = 1. \tag{9}$$

Let $x = (x_1, \ldots, x_k)$. Let $G(x) = \sum_{t=1}^k x_t C_t$ and consider the polynomial

$$g(x) = D_\omega\left(G(x)\right) = \sum_{\sigma \in \mathbb{S}_k} \omega(\sigma) \prod_{\ell=1}^k G(x)(\ell, \sigma(\ell)) \in \mathbb{F}[x_1, \ldots, x_k].$$

Claim 2.4. *There exists a $\lambda \in \mathbb{F}^k$ such that $g(\lambda) \neq 0$.*

We will use Alon's Combinatorial Nullstellensatz (Theorem 1.2 in [1]).

Theorem 2.5 (Alon [1])**.** *Let $\mathbb{F}$ be an be an arbitrary field and let $g = g(x_1, \ldots, x_k) \in \mathbb{F}[x_1 \ldots, x_k]$. Suppose the total degree* $\deg(g)$ *of g is $\sum_{t=1}^k d_t$ where each d_t is a nonnegative integer, and suppose the coefficient of $\prod_{t=1}^k x_t^{d_t}$ in g is nonzero. Then, if $\Lambda_1, \ldots, \Lambda_k$ are subsets of $\mathbb{F}$ with $|\Lambda_t| > d_t$, there exist $\lambda_1 \in \Lambda_1, \ldots, \lambda_k \in \Lambda_t$ such that $g(\lambda_1, \ldots, \lambda_k) \neq 0$.*

Proof of Claim 2.4: We will show that the monomial $x_1 \cdots x_k$ appears with a nonzero coefficient in $g(x)$. Indeed, let $\sigma \in \mathbb{S}_k$. By (7), for any $1 \leq t \leq k$ the variable x_t does not appear in $\prod_{\ell<t} G(x)\,(\ell, \sigma(\ell))$. It follows that the coefficient of $x_1 \cdots x_k$ in $\prod_{\ell=1}^k G(x)\,(\ell, \sigma(\ell))$ is

$$\gamma(\sigma) := \prod_{\ell=1}^k C_\ell(\ell, \sigma(\ell)).$$

Let $1 \leq \ell \leq k$. If $\sigma(\ell) < \pi^{-1}(\ell)$ then $C_\ell(\ell, \sigma(\ell)) = 0$ by (8). Therefore if $\gamma(\sigma) \neq 0$ then $\sigma(\ell) \geq \pi^{-1}(\ell)$ for all $1 \leq \ell \leq k$, i.e. $\sigma = \pi^{-1}$. Together with (9), it follows that the coefficient of $x_1 \cdots x_k$ in $g(x)$ is

$$\omega(\pi^{-1})\gamma(\pi^{-1}) = \omega(\pi^{-1}) \prod_{\ell=1}^k C_\ell(\ell, \pi^{-1}(\ell)) = \omega(\pi^{-1}) \neq 0.$$

Applying Theorem 2.5 for the degree k polynomial g, with $d_1 = \cdots = d_k = 1$ and $\Lambda_1 = \cdots = \Lambda_k = \{0, 1\}$, it follows that there exists $\lambda \in \{0, 1\}^k$ such that $D_\omega\left(G(\lambda)\right) = g(\lambda) \neq 0$. As $G(\lambda)$ is a $k \times k$ submatrix of $\sum_{t=1}^k \lambda_t A_t \in W$ it follows that $\rho_\omega(W) \geq k$.

□

Proof of Theorem 1.5. The $\geq$ direction of (3) follows from Claim 2.1. Indeed, if $\mathcal{B}' \subset \mathcal{B}$ satisfies $\nu_b(\mathcal{B}') \leq k$, then $W = M_{\mathcal{B}'}(\mathbb{F}) \leq M_{\mathcal{B}}(\mathbb{F})$ satisfies $\rho_\omega(W) = \nu_b(\mathcal{B}') \leq k$ and $\dim W = |\mathcal{B}'|$. For the $\leq$ direction, suppose $W \leq M_{\mathcal{B}}(\mathbb{F})$ satisfies $\rho_\omega(W) \leq k$. Let $\mathcal{B}' = \mathbb{B}(W) \subset \mathcal{B}$. Then $|\mathcal{B}'| = \dim W$ and Proposition 2.2 implies that $\nu_b(\mathcal{B}') \leq \rho_\omega(W) \leq k$.

□

3 Maximal Dimensional Spaces of Bounded Permanental Rank

In this section we prove Theorem 1.8. The main ingredient of the argument is the following

Proposition 3.1. *Let $k \geq 2$ and suppose that* $\operatorname{char} \mathbb{F} \neq 2$ *and $U \leq M_{k+1}(\mathbb{F})$ satisfies $\mathbb{B}(U) = [k] \times [k+1]$ and $\rho_{\mathbb{1}}(U) = k$. Then $U = \operatorname{span}\{e_i\}_{i \in [k]} \otimes \mathbb{F}^{k+1}$.*

Proof. The assumption $\mathbb{B}(U) = [k] \times [k+1]$ implies that U has a basis

$$\{A_{ij} : (i,j) \in [k] \times [k+1]\}$$

such that

$$A_{ij} = e_i \otimes e_j + \sum_{\ell=1}^{k+1} \lambda_{ij\ell}(e_{k+1} \otimes e_\ell)$$

for some $\lambda_{ij\ell} \in \mathbb{F}$.

Claim 3.2. $\lambda_{ij\ell} = 0$ *for any* $1 \leq i \leq k$ *and* $1 \leq j \neq \ell \leq k+1$.

Proof. Suppose for contradiction that $\lambda_{ij\ell} \neq 0$ for some $1 \leq i \leq k$ and $1 \leq j \neq \ell \leq k+1$. Let

$$[k] \setminus \{i\} = \{i_1, \ldots, i_{k-1}\} \quad , \quad [k+1] \setminus \{\ell, j\} = \{j_1, \ldots, j_{k-1}\},$$

and for $\theta \in \mathbb{F}$ let

$$C_\theta = \theta A_{ij} + \sum_{t=1}^{k-1} A_{i_t j_t} \in U.$$

Clearly, the only permutation $\pi \in \mathbb{S}_{k+1}$ that corresponds to a nonzero term in the expansion of $\operatorname{per} C_\theta$ is that given by $\pi(i) = j, \pi(k+1) = \ell$ and $\pi(i_t) = j_t$ for $1 \leq t \leq k-1$. Therefore

$$\begin{aligned}
\operatorname{per} C_\theta &= C_\theta(i,j) \cdot C_\theta(k+1,\ell) \cdot \prod_{t=1}^{k-1} C_\theta(i_t, j_t) \\
&= C_\theta(i,j) \cdot C_\theta(k+1,\ell) \\
&= \theta \left(\theta A_{ij}(k+1,\ell) + \sum_{t=1}^{k-1} A_{i_t j_t}(k+1,\ell) \right) \\
&= \theta \left(\theta \lambda_{ij\ell} + \sum_{t=1}^{k-1} \lambda_{i_t j_t \ell} \right).
\end{aligned} \tag{10}$$

Eq. (10) and the assumption $\lambda_{ij\ell} \neq 0$ imply that $\operatorname{per} C_\theta$ is a nonzero polynomial of degree 2 in θ. As $|\mathbb{F}| \geq 3$ there exists $\theta \in \mathbb{F}$ such that $\operatorname{per} C_\theta \neq 0$, contradicting $\rho_{\mathbb{1}}(U) = k$.

□

For $(i,j) \in [k] \times [k+1]$ let $\mu_{ij} = \lambda_{ijj}$. Claim 3.2 implies that $A_{ij} = e_i \otimes e_j + \mu_{ij} e_{k+1} \otimes e_j$.

Claim 3.3. $\mu_{ij} = 0$ *for all* $(i,j) \in [k] \times [k+1]$.

Proof. Fix $i \in [k]$ and $j' \neq j'' \in [k+1]$. Let

$$[k] \setminus \{i\} = \{i_1, \ldots, i_{k-1}\} \quad , \quad [k+1] \setminus \{j', j''\} = \{j_1, \ldots, j_{k-1}\},$$

and let

$$C = A_{ij'} + A_{ij''} + \sum_{t=1}^{k-1} A_{i_t j_t}.$$

Clearly, if $\pi \in \mathbb{S}_{k+1}$ corresponds to a nonzero term in the expansion of $\operatorname{per} C$ then $\pi(i_t) = j_t$ for $1 \leq t \leq k-1$ and either $(\pi(i), \pi(k+1)) = (j', j'')$ or $(\pi(i), \pi(k+1)) = (j'', j')$. It follows that

$$\begin{aligned} \operatorname{per} C &= \big(C(i,j') \cdot C(k+1,j'') + C(i,j'') \cdot C(k+1,j')\big) \cdot \prod_{t=1}^{k-1} C(i_t, j_t) \\ &= C(k+1, j'') + C(k+1, j') = \mu_{ij''} + \mu_{ij'}. \end{aligned}$$

Together with the assumption that $\rho_{\mathbb{1}}(U) = k$ this implies that

$$\mu_{ij''} + \mu_{ij'} = \operatorname{per} C = 0. \tag{11}$$

As (11) holds for all $1 \leq j' \neq j'' \leq k+1$ and $k \geq 2$, the assumption $\operatorname{char}(\mathbb{F}) \neq 2$ implies that $\mu_{ij} = 0$ for all $(i,j) \in [k] \times [k+1]$, thereby completing the proof of Claim 3.2 and of Proposition 3.1.

□

Proof of Theorem 1.8. We may assume that $k < n$. Let $W \leq M_n(\mathbb{F})$ satisfy $\rho_{\mathbb{1}}(W) = k$ and $\dim W = kn$. Proposition 2.2 implies that

$$\nu(\mathbb{B}(W)) \leq \rho_{\mathbb{1}}(W) = k.$$

As $|\mathbb{B}(W)| = \dim W = kn$, it follows by Kőnig's theorem that $\mathbb{B}(W)$ is either $I \times [n]$ or $[n] \times I$ for some $I \in \binom{[n]}{k}$. Consider the first case $\mathbb{B}(W) = I \times [n]$ and let $I = \{i_1, \ldots, i_k\}$ where $1 \leq i_1 < \cdots < i_k \leq n$. We have to show that $W = \operatorname{span}\{e_i : i \in I\} \otimes \mathbb{F}^n$. Fix a pair $(i', j') \in ([n] \setminus I) \times [n]$. Define $i_{k+1} = i'$ and choose $J = \{j_1, \ldots, j_{k+1}\} \subset [n]$ such that $j_1 < \cdots < j_{k+1}$ and $j' = j_{\beta'}$ for some $1 \leq \beta' \leq k+1$. For $A \in W$ let $\tilde{A} \in M_{k+1}(\mathbb{F})$ be given by $\tilde{A}(\alpha, \beta) = A(i_\alpha, j_\beta)$ for $(\alpha, \beta) \in [k+1]^2$. Let $U = \{\tilde{A} : A \in W\}$. Clearly $\mathbb{B}(U) = [k] \times [k+1]$ and $\rho_{\mathbb{1}}(U) = k$. Proposition 3.1 then implies that $U = \operatorname{span}\{e_\alpha\}_{\alpha \in [k]} \otimes \mathbb{F}^{k+1}$. In particular $A(i', j') = \tilde{A}(k+1, \beta') = 0$ for all $A \in W$. Therefore

$$W \leq \operatorname{span}\{e_i\}_{i \in I} \otimes \mathbb{F}^n. \tag{12}$$

As $\dim W = kn$ it follows that there is equality in (12). The case $\mathbb{B}(W) = [n] \times I$ for some $|I| = k$ is handled similarly.

□

4 Maximal Rank in Subspaces of $A_{\mathcal{G}}(\mathbb{F})$

In this section we prove Theorem 1.9. We first recall the definition of the *Pfaffian* of an alternating matrix of even order $A \in A_{2k}(\mathbb{F})$. A *perfect matching* in K_{2k} is a matching of size k. Let $\mathcal{M}_{2k}$ denote the set of all perfect matchings in K_{2k}. For $M = \{f_1, \ldots, f_k\} \in \mathcal{M}_{2k}$ such that $f_t = \{i_t < j_t\}$ for $1 \le t \le k$ and $i_1 < \cdots < i_k$, let

$$\theta(M) = \operatorname{sgn} \begin{pmatrix} 1 & 2 & \cdots & 2k-1 & 2k \\ i_1 & j_1 & \cdots & i_k & j_k \end{pmatrix}$$

and let

$$\mu(A, M) = \prod_{t=1}^{k} A(i_t, j_t).$$

The Pfaffian of A is defined by

$$\operatorname{Pf}(A) = \sum_{M \in \mathcal{M}_{2k}} \theta(M)\mu(A, M).$$

It is well known that $\det(A) = \operatorname{Pf}(A)^2$ (see e.g. Exercise 4.24 in [7]). We will need the following facts.

Claim 4.1. *For any $\mathcal{G} \subset \binom{[n]}{2}$*

$$\rho(A_{\mathcal{G}}(\mathbb{F})) = 2\nu(\mathcal{G}). \tag{13}$$

Proof. Let $\rho(A_{\mathcal{G}}(\mathbb{F})) = 2k$ and $\nu(\mathcal{G}) = \ell$. Let $\{\{i_1, j_1\}, \ldots, \{i_\ell, j_\ell\}\}$ be a matching in $\mathcal{G}$. Then $A = \sum_{t=1}^{\ell} e_{i_t} \wedge e_{j_t} \in A_{\mathcal{G}}(\mathbb{F})$ and $\operatorname{rk}(A) = 2\ell$. Therefore $k \ge \ell$. For the other direction, let $A \in A_{\mathcal{G}}(\mathbb{F})$ such that $\operatorname{rk}(A) = 2k$. Then there exists $I = \{\alpha_1 < \cdots < \alpha_{2k}\}$ such that $B = A[I|I] \in A_{2k}(\mathbb{F})$ is nonsingular. Thus

$$0 \ne \det(B) = \operatorname{Pf}(B)^2 = \left(\sum_{M \in \mathcal{M}_{2k}} \theta(M)\mu(B, M) \right)^2.$$

Hence there exists a matching $M = \{f_1, \ldots, f_k\} \in \mathcal{M}_{2k}$ such that $\mu(B, M) \ne 0$. Writing $f_t = \{i_t < j_t\}$ for $1 \le t \le k$ it follows that

$$0 \ne \mu(B, M) = \prod_{t=1}^{k} B(i_t, j_t) = \prod_{t=1}^{k} A(\alpha_{i_t}, \alpha_{j_t}).$$

Thus $\{\{\alpha_{i_t}, \alpha_{j_t}\} : 1 \le t \le k\}$ is a matching of size k in $\mathcal{G}$ and therefore $\ell = \nu(\mathcal{G}) \ge k$.

□

For $A \in A_n(\mathbb{F})$ with $q(A) = (i, j)$ define $\tilde{q}(A) = \{i, j\}$. For $S \subset A_n(\mathbb{F})$ let $\mathbb{G}(S) = \{\tilde{q}(A) : A \in S\}$. Note that if $U \le A_n(\mathbb{F})$ is a linear subspace then $|\mathbb{G}(U)| = \dim U$. The key ingredient in the proof of Theorem 1.9 is the following result (Theorem 1.2 in [9]).

Theorem 4.2 ([9])**.** *Let $U \le A_n(\mathbb{F})$ be a linear subspace. Then $\rho(U) \ge 2\nu(\mathbb{G}(U))$.*

Proof of Theorem 1.9. The $\ge$ direction of (4) follows from Claim 4.1. Indeed, if $\mathcal{G}' \subset \mathcal{G}$ satisfies $\nu(\mathcal{G}') \le k$, then $U = A_{\mathcal{G}'}(\mathbb{F}) \le A_{\mathcal{G}}(\mathbb{F})$ satisfies $\rho(U) = 2\nu(\mathcal{G}') \le 2k$ and $\dim U = |\mathcal{G}'|$. For the $\le$ direction, suppose $U \le A_{\mathcal{G}}(\mathbb{F})$ satisfies $\rho(U) \le 2k$. Let $\mathcal{G}' = \mathbb{G}(U) \subset \mathcal{G}$. Then $|\mathcal{G}'| = \dim U$ and Theorem 4.2 implies that $\nu(\mathcal{G}') \le \frac{\rho(U)}{2} \le k$.

□

5 Concluding Remarks

In this note we proved two extensions of the combinatorial characterization due to Li, Qiao, Wigderson, Wigderson and Zhang [6] of the maximal dimension of bounded rank subspaces of the graphical matrix space $M_{\mathcal{B}}(\mathbb{F})$ associated with a bipartite graph $\mathcal{B}$. Theorem 1.5 shows that the above characterization remains valid for a wide class of generalized rank functions, including e.g. the permanental rank. In a different direction, Theorem 1.9 extends the characterization to bounded rank subspaces of the graphical matrix space $A_{\mathcal{G}}(\mathbb{F})$ associated with a general graph $\mathcal{G}$. We conclude with the following two remarks.

- Theorem 1.1 provides a classification of the spaces $W \le M_n(\mathbb{F})$ such that $\rho(W) = k$ and $\dim W = kn$. The analogous (but different) classification of spaces $W \le M_n(\mathbb{F})$ such that $\rho_{\mathbb{1}}(W) = k$ and $\dim W = kn$ is given in Theorem 1.8. It would be interesting to extend these results to bounded rank subspaces of $M_{\mathcal{B}}(\mathbb{F})$ and of $A_{\mathcal{G}}(\mathbb{F})$ for various $\mathcal{B} \subset [n]^2$ and $\mathcal{G} \subset \binom{[n]}{2}$.
- For an element u in the p-th exterior power $\bigwedge^p \mathbb{F}^n$, let $E(u)$ denote the minimal subspace $U \le \mathbb{F}^n$ such that $u \in \bigwedge^p U$. The *rank* of u is $\text{rk}(u) = \dim E(u)$. For sufficiently large fields, Theorem 4.2 is the case $p = 2$ of Theorem 2.1 in [3] that gives a lower bound on $\rho(U)$ for $U \le \bigwedge^p \mathbb{F}^n$ in terms of the weak matching number of a certain p-uniform hypergraph associated to U. It seems likely that this lower bound may be useful in obtaining extensions of Theorem 1.9 to structured subspaces of $\bigwedge^p \mathbb{F}^n$ for general p.

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