

# Random Balanced Cayley Complexes

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## Abstract

Let  $G$  be a finite group of order  $n$  and for  $1 \leq i \leq k+1$  let  $V_i = \{i\} \times G$ . Viewing each  $V_i$  as a 0-dimensional complex, let  $Y_{G,k}$  denote the simplicial join  $V_1 * \cdots * V_{k+1}$ . For  $A \subset G$  let  $Y_{A,k}$  be the subcomplex of  $Y_{G,k}$  that contains the  $(k-1)$ -skeleton of  $Y_{G,k}$  and whose  $k$ -simplices are all  $\{(1, x_1), \dots, (k+1, x_{k+1})\} \in Y_{G,k}$  such that  $x_1 \cdots x_{k+1} \in A$ . Let  $L_{k-1}$  denote the reduced  $(k-1)$ -th Laplacian of  $Y_{A,k}$ , acting on the space  $C^{k-1}(Y_{A,k})$  of real valued  $(k-1)$ -cochains of  $Y_{A,k}$ . The  $(k-1)$ -th spectral gap  $\mu_{k-1}(Y_{A,k})$  of  $Y_{A,k}$  is the minimal eigenvalue of  $L_{k-1}$ .

The following  $k$ -dimensional analogue of the Alon-Roichman theorem is proved: Let  $k \geq 1$  and  $\epsilon > 0$  be fixed and let  $A$  be a random subset of  $G$  of size  $m = \left\lceil \frac{9k^2 \log D}{\epsilon^2} \right\rceil$  where  $D$  is the sum of the degrees of the complex irreducible representations of  $G$ . Then

$$\Pr[\mu_{k-1}(Y_{A,k}) < (1 - \epsilon)m] = O\left(\frac{1}{n}\right).$$

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## 1 Introduction

The Laplacian  $L(\mathcal{C})$  of a graph  $\mathcal{C} = (V, E)$  is the  $V \times V$  positive semidefinite matrix whose  $(u, v)$  entry is given by

$$L(\mathcal{C})_{uv} = \begin{cases} \deg_{\mathcal{C}}(u) & u = v, \\ -1 & \{u, v\} \in E, \\ 0 & \text{otherwise.} \end{cases}$$

Let  $0 = \lambda_1(\mathcal{C}) \leq \lambda_2(\mathcal{C}) \leq \cdots \leq \lambda_{|V|}(\mathcal{C})$  be the eigenvalues of  $L(\mathcal{C})$ . The second smallest eigenvalue  $\lambda_2(\mathcal{C})$ , called the *spectral gap* of  $\mathcal{C}$ , is a parameter of central importance in a variety of problems. In particular it controls the expansion properties of  $\mathcal{C}$  and the convergence rate of a random walk on  $\mathcal{C}$  (see e.g. chapters XIII and IX in [5]).

Throughout the paper, let  $G$  denote a finite group of order  $n$  and let  $\widehat{G} = \{\rho\}$  be the set of irreducible unitary representations of  $G$ , where  $\rho : G \rightarrow U(d_\rho)$ . Let  $D(G) = \sum_{\rho \in \widehat{G}} d_\rho$ . Let  $\mathbf{1} \in \widehat{G}$  denote the trivial representation of  $G$  and let  $\widehat{G}_+ = \widehat{G} \setminus \{\mathbf{1}\}$ .

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Let  $T \subset G$  be symmetric subset, i.e.  $T = T^{-1}$ . The *Cayley graph*  $\mathcal{C}(G, T)$  of  $G$  with respect to  $T$  is the graph on the vertex set  $G$  with edge set  $\{\{g, gt\} : g \in G, t \in T\}$ . The seminal Alon-Roichman theorem [1] is concerned with the expansion of Cayley graphs with respect to random sets of generators.

**Theorem 1.1** (Alon-Roichman). *For any  $\epsilon > 0$  there exists a constant  $c(\epsilon) > 0$  such that for any group  $G$ , if  $S$  is a random subset of  $G$  of size  $\lceil c(\epsilon) \log |G| \rceil$  and  $m = |S \cup S^{-1}|$ , then  $\lambda_2(\mathcal{C}(G, S \cup S^{-1}))$  is asymptotically almost surely (a.a.s.) at least  $(1 - \epsilon)m$ .*

**Remark 1.2.** *Landau and Russell [10] and independently Loh and Schulman [12] have obtained an improvement on Theorem 1.1 by showing that the  $\log |G|$  factor in the bound on  $|S|$  can be replaced by  $\log D(G)$ . While this does not change the logarithmic dependence of  $|S|$  on  $|G|$ , it does often lead to an improvement of the constant  $c(\epsilon)$ .*

This paper is concerned with higher dimensional counterparts of Theorem 1.1. We briefly recall the relevant terminology (see section 2 for details). For a simplicial complex  $X$  and  $k \geq -1$  let  $X^{(k)}$  denote the  $k$ -dimensional skeleton of  $X$ . For  $k \geq -1$  let  $C^k(X)$  denote the space of real valued simplicial  $k$ -cochains of  $X$  and let  $d_k : C^k(X) \rightarrow C^{k+1}(X)$  denote the coboundary operator. For  $k \geq 0$  define the reduced  $k$ -th Laplacian of  $X$  by  $L_k(X) = d_{k-1}d_{k-1}^* + d_k^*d_k$ . The minimal eigenvalue of  $L_k(X)$ , denoted by  $\mu_k(X)$ , is the  $k$ -th spectral gap of  $X$ .

The following  $k$ -dimensional abelian version of Theorem 1.1 was obtained in [3]. Let  $H$  be an additively written abelian group of order  $h$  and let  $k \leq h$ . Let  $\Delta_{h-1}$  denote the  $(h-1)$ -simplex on the vertex set  $H$ . The *Sum Complex*  $X_{A,k}$  associated with a subset  $A \subset H$  is the  $k$ -dimensional simplicial complex obtained by taking the full  $(k-1)$ -skeleton of  $\Delta_{h-1}$  together with all  $(k+1)$ -subsets  $\sigma \subset H$  that satisfy  $\sum_{x \in \sigma} x \in A$ .

**Theorem 1.3** ([3]). *Let  $k \geq 1$  and  $\epsilon > 0$  be fixed and let  $A$  be a random subset of  $H$  of size  $m = \lceil \frac{4k^2 \log h}{\epsilon^2} \rceil$ . Then*

$$\Pr[\mu_{k-1}(X_{A,k}) < (1 - \epsilon)m] = O\left(\frac{1}{n}\right).$$

**Remark 1.4.** *See [11, 15] for more on sum complexes and their cohomology.*

In the present paper we study a different model of Cayley complexes associated with subsets of general finite groups and obtain a new high dimensional analogue of Theorem 1.1. Recall that  $G$  is a finite group of order  $n$  and let  $k \geq 1$ . For  $1 \leq i \leq k+1$  let  $V_i = \{i\} \times G$ . Let  $Y_{G,k}$  denote the simplicial join  $V_1 * \cdots * V_{k+1}$ , where each  $V_i$  is viewed as 0-dimensional complex. Thus  $Y_{G,k}$  is homotopy equivalent to an  $N$ -fold wedge  $\bigvee^N S^k$  of  $k$ -dimensional spheres, where  $N = (n-1)^{k+1}$ . The *balanced  $k$ -dimensional Cayley Complex* associated with a subset  $\emptyset \neq A \subset G$  is the subcomplex  $Y_{A,k} \subset Y_{G,k}$  whose  $k$ -simplices are all  $\{(1, y_1), \dots, (k+1, y_{k+1})\} \in Y_{G,k}$  such that  $y_1 \cdots y_{k+1} \in A$ . Note that  $Y_{A,k} \supset Y_{G,k}^{(k-1)}$ .

Let  $1_A$  denote the indicator function of  $A \subset G$ , i.e.  $1_A(x) = 1$  if  $x \in A$  and  $1_A(x) = 0$  otherwise. For a representation  $\rho : G \rightarrow GL_d(\mathbb{C})$  let  $\widehat{1_A}(\rho) = \sum_{x \in A} \rho(x) \in M_d(\mathbb{C})$  be the Fourier transform of  $1_A$  at  $\rho$  (see section 4 for details). For a matrix  $T \in M_d(\mathbb{C})$  let  $\|T\| = \max_{\|v\|=1} \|Tv\|$  denote the spectral norm of  $T$ . Let  $\nu(A) = \max_{\rho \in \widehat{G}_+} \|\widehat{1_A}(\rho)\|$ . Our first result is a lower bound on  $\mu_{k-1}(Y_{A,k})$  in terms of  $\nu(A)$ .

**Theorem 1.5.**

$$\mu_{k-1}(Y_{A,k}) \geq |A| - k \cdot \nu(A).$$

Our main result is the following  $k$ -dimensional analogue of the Alon-Roichman Theorem.

**Theorem 1.6.** *Let  $k$  and  $\epsilon > 0$  be fixed. Let  $G$  be a finite group of order  $n$  and fix an integer  $m$  such that  $\frac{9k^2 \log D(G)}{\epsilon^2} \leq m \leq \sqrt{n}$ . Let  $A$  be a random subset of  $G$  of size  $m$ . Then*

$$\Pr[\mu_{k-1}(Y_{A,k}) < (1 - \epsilon)m] < \frac{6}{n}.$$

**Remark 1.7.** (i) *It is straightforward to check that  $\mu_{k-1}(Y_{A,k}) \leq |A| + k$  for any  $A \subset G$  (see Eq. (2) in [3]). Theorem 1.6 thus implies that if  $A$  is a random subset of  $G$  and  $\log |G| = o(|A|)$ , then  $Y_{A,k}$  is a.a.s. a near optimal spectral expander. (ii) While there are some similarities between sum complexes and balanced Cayley complexes, the analysis of  $Y_{A,k}$  in the present paper requires some additional ideas, including the use of the non-abelian Fourier transform and of Garland's eigenvalue estimates [8].*

Our final result concerns the homotopy type of  $Y_{A,k}$  when  $A$  is a subgroup of  $G$ . For  $1 \leq m$  let

$$\begin{aligned} \gamma_0(m, k) &= (n - m)n^k + \binom{n}{m}^k (m - 1)^{k+1} - (n - 1)^{k+1}, \\ \gamma_1(m, k) &= \binom{n}{m}^k (m - 1)^{k+1}. \end{aligned}$$

**Theorem 1.8.** *Let  $A$  be a subgroup of  $G$  of order  $|A| = m$ . Then*

(i)

$$Y_{A,1} \simeq \coprod_{n/m} \bigvee_{(m-1)^2} S^1. \tag{1}$$

(ii) For  $k \geq 2$

$$Y_{A,k} \simeq \bigvee_{\gamma_0(m,k)} S^{k-1} \vee \bigvee_{\gamma_1(m,k)} S^k. \tag{2}$$

**Remark 1.9.** *As  $\gamma_0(m, k) > 0$  for all  $m < n$ , it follows from Theorem 1.8 that if  $A \subset G$  generates a subgroup  $\langle A \rangle$  of order  $m < n$  then*

$$\tilde{\beta}_{k-1}(Y_{A,k}) \geq \tilde{\beta}_{k-1}(Y_{\langle A \rangle, k}) = \gamma_0(m, k) > 0$$

*and therefore  $\mu_{k-1}(Y_{A,k}) = 0$ . As there are families of groups  $G$  (e.g. elementary abelian groups of fixed exponent) that cannot be generated by  $o(\log |G|)$  elements, this implies that the  $\log D(G) = \Theta(\log n)$  factor in Theorem 1.6 cannot in general be improved.*

The paper is organized as follows. In Section 2 we review some basic properties of high dimensional Laplacians and their eigenvalues, including Garland's lower bound for the higher spectral gaps. In Section 3 we compute the spectra of various Laplacians of the skeleta of  $Y_{G,k}$  and deduce a variational characterization (Proposition 3.1) of  $\mu_{k-1}(Y)$  for subcomplexes  $Y_{G,k}^{(k-1)} \subset Y \subset Y_{G,k}$ . In Section 4 we briefly recall the definition and some basic properties of the Fourier transform on finite groups. In Section 5 we prove Theorem 1.5. This bound is the key ingredient in the proof of Theorem 1.6 given in Section 6. In Section 7 we prove Theorem 1.8. We conclude in Section 8 with some remarks and open problems.

## 2 Laplacians and their Eigenvalues

Let  $X$  be a finite simplicial complex on the vertex set  $V$ . Let  $X(k)$  denote the set of  $k$ -dimensional simplices in  $X$ , each taken with an arbitrary but fixed orientation. A simplicial  $k$ -cochain is a real valued skew-symmetric function on all ordered  $k$ -simplices of  $X$ . For  $k \geq 0$  let  $C^k(X)$  denote the space of  $k$ -cochains on  $X$ . The  $i$ -face of an ordered  $(k+1)$ -simplex  $\sigma = [v_0, \dots, v_{k+1}]$  is the ordered  $k$ -simplex  $\sigma_i = [v_0, \dots, \widehat{v}_i, \dots, v_{k+1}]$ . The coboundary operator  $d_k : C^k(X) \rightarrow C^{k+1}(X)$  is given by

$$d_k \phi(\sigma) = \sum_{i=0}^{k+1} (-1)^i \phi(\sigma_i) .$$

It will be convenient to augment the cochain complex  $\{C^i(X)\}_{i \geq 0}$  with the  $(-1)$ -degree term  $C^{-1}(X) = \mathbb{C}$  with the coboundary map  $d_{-1} : C^{-1}(X) \rightarrow C^0(X)$  given by  $d_{-1}(a)(v) = a$  for  $a \in \mathbb{C}$ ,  $v \in V$ . Let  $Z^k(X) = \ker d_k$  denote the space of  $k$ -cocycles and let  $B^k(X) = \text{Im } d_{k-1}$  denote the space of  $k$ -coboundaries. For  $k \geq 0$  let  $\tilde{H}^k(X) = Z^k(X)/B^k(X)$  denote the  $k$ -th reduced cohomology group of  $X$  with real coefficients. For each  $k \geq -1$  endow  $C^k(X)$  with the standard inner product  $(\phi, \psi)_X = \sum_{\sigma \in X(k)} \phi(\sigma)\psi(\sigma)$  and the corresponding  $L^2$  norm  $\|\phi\|_X = (\phi, \phi)_X^{1/2}$ . Let  $d_k^* : C^{k+1}(X) \rightarrow C^k(X)$  denote the adjoint of  $d_k$  with respect to these standard inner products. The reduced  $k$ -th lower and upper Laplacians of  $X$  are the positive semidefinite self-adjoint maps of  $C^k(X)$  given respectively by  $L_k^-(X) = d_{k-1}d_{k-1}^*$  and  $L_k^+(X) = d_k^*d_k$ . The  $k$ -th Laplacian of  $X$  is  $L_k(X) = L_k^-(X) + L_k^+(X)$ . Let  $\mathcal{H}^k(X) = \ker L_k(X) = \ker d_{k-1}^* \cap \ker d_k$  denote the space of harmonic  $k$ -cochains. When there is no ambiguity concerning  $X$ , we shall abbreviate  $L_k(X) = L_k$  and  $L_k^\pm(X) = L_k^\pm$ . Clearly

$$L_k^-(\text{Im } d_{k-1}) \subset \text{Im } d_{k-1} \quad , \quad L_k^+(\text{Im } d_k^*) \subset \text{Im } d_k^* .$$

For a self-adjoint map  $T$  on an inner product space  $W$  let  $\mathcal{S}(W, T)$  denote the set of eigenvalues of  $T$  and let  $s(W, T, \lambda)$  denote the multiplicity of an eigenvalue  $\lambda \in \mathcal{S}(W, T)$ . Let  $\tilde{\mathcal{S}}(W, T)$  denote the multiset consisting of  $s(W, T, \lambda)$  copies of each  $\lambda \in \mathcal{S}(W, T)$ . The  $k$ -th spectral gap of  $X$  is

$$\mu_k(X) = \min \mathcal{S} \left( C^k(X), L_k \right) .$$

**Remark 2.1.** (i) If  $X$  is a graph then  $\mu_0(X) = \lambda_2(X)$ . (ii)  $\mu_k(X) > 0$  iff  $\tilde{H}^k(X; \mathbb{R}) = 0$ , hence  $\mu_k$  may be viewed as a robustness measure of the property of having vanishing  $k$ -dimensional real cohomology.

The lower and upper  $k$ -th spectral gaps of  $X$  are defined respectively by

$$\mu_k^-(X) = \min \mathcal{S} \left( \text{Im } d_{k-1}, L_k^- \right)$$

and

$$\mu_k^+(X) = \min \mathcal{S} \left( \text{Im } d_k^*, L_k^+ \right) .$$

In Section 3 we will use some well known facts concerning Laplacians and their eigenvalues. For proofs, see e.g. [6].

**Proposition 2.2.** Let  $0 \leq k \leq \dim X$ . Then the following hold:

(i) *Hodge Decomposition: There is an orthogonal direct sum decomposition:*

$$C^k(X) = \text{Im } d_{k-1} \oplus \mathcal{H}^k(X) \oplus \text{Im } d_k^* \quad (3)$$

(ii)

$$\ker L_k^- = \mathcal{H}^k(X) \oplus \text{Im } d_k^* \quad , \quad \ker L_k^+ = \text{Im } d_{k-1} \oplus \mathcal{H}^k(X). \quad (4)$$

(iii) *Hodge isomorphism:*

$$\mathcal{H}^k(X) \cong \tilde{H}^k(X). \quad (5)$$

(iv) *For all  $\lambda \neq 0$*

$$s\left(C^k(X), L_k, \lambda\right) = s\left(\text{Im } d_{k-1}, L_k^-, \lambda\right) + s\left(\text{Im } d_k^*, L_k^+, \lambda\right) \quad (6)$$

(v)

$$\tilde{\mathcal{S}}\left(\text{Im } d_{k-1}^*, L_{k-1}^+\right) = \tilde{\mathcal{S}}\left(\text{Im } d_{k-1}, L_k^-\right). \quad (7)$$

(vi) *If  $H^k(X) = 0$  then*

$$\mu_k(X) = \min \left\{ \mu_k^-(X), \mu_k^+(X) \right\}.$$

In section 5 we shall use the following special case of Garland's fundamental eigenvalue estimate (see Section 5 of [8] and Theorem 1.12 of [2]). The link of a simplex  $\tau \in X(\ell)$  is  $X_\tau = \text{lk}(X, \tau) = \{\eta \in X : \tau \cap \eta = \emptyset, \tau \cup \eta \in X\}$ . For  $\phi \in C^j(X)$  and  $\tau \in X(\ell)$  let  $\phi_\tau \in C^{j-\ell-1}(X_\tau)$  be defined by  $\phi_\tau(\eta) = \phi(\eta\tau)$ , where  $\eta\tau$  denotes the concatenation of  $\eta$  and  $\tau$ .

**Theorem 2.3** (Garland [8]). *Let  $X$  be a  $k$ -dimensional complex such that for all  $\sigma \in X(k-1)$*

$$\deg_X(\sigma) := |\{\eta \in X(k) : \sigma \subset \eta\}| = m.$$

*Let  $\lambda(X) = \min \{\lambda_2(X_\tau) : \tau \in X(k-2)\}$ . Then*

$$\min \left\{ \frac{\|d_{k-1}\phi\|_X^2}{\|\phi\|_X^2} : 0 \neq \phi \in \ker d_{k-2}^* \right\} \geq k\lambda(X) - (k-1)m.$$

For completeness we indicate the proof. We first establish the following identity.

**Claim 2.4.** *For any  $\phi \in C^{k-1}(X)$*

$$\|d_{k-1}\phi\|_X^2 = \sum_{\tau \in X(k-2)} \|d_0\phi_\tau\|_{X_\tau}^2 - m(k-1)\|\phi\|_X^2. \quad (8)$$

**Proof.**

$$\begin{aligned}
\|d_{k-1}\phi\|_X^2 &= \sum_{\sigma \in X(k)} |d_{k-1}\phi(\sigma)|^2 \\
&= \sum_{\sigma \in X(k)} \left( \sum_{i=0}^k (-1)^i \phi(\sigma_i) \right) \left( \sum_{j=0}^k (-1)^j \phi(\sigma_j) \right) \\
&= \sum_{\sigma \in X(k)} \sum_{i=0}^k \phi(\sigma_i)^2 + 2 \sum_{\sigma \in X(k)} \sum_{0 \leq i < j \leq k} (-1)^{i+j} \phi(\sigma_i) \phi(\sigma_j) \\
&= m \|\phi\|_X^2 - 2 \sum_{\tau \in X(k-2)} \sum_{uv \in X_\tau(1)} \phi(u\tau) \phi(v\tau).
\end{aligned} \tag{9}$$

On the other hand

$$\begin{aligned}
\sum_{\tau \in X(k-2)} \|d_0\phi_\tau\|_{X_\tau}^2 &= \sum_{\tau \in X(k-2)} \sum_{uv \in X_\tau(1)} (\phi(v\tau) - \phi(u\tau))^2 \\
&= \sum_{\tau \in X(k-2)} \sum_{uv \in X_\tau(1)} (\phi(u\tau)^2 + \phi(v\tau)^2) - 2 \sum_{\tau \in X(k-2)} \sum_{uv \in X_\tau(1)} \phi(u\tau) \phi(v\tau) \\
&= mk \|\phi\|_X^2 - 2 \sum_{\tau \in X(k-2)} \sum_{uv \in X_\tau(1)} \phi(u\tau) \phi(v\tau).
\end{aligned} \tag{10}$$

Subtracting (10) from (9) we obtain (8). □

**Proof of Theorem 2.3.** Let  $\phi \in \ker d_{k-2}^*$ . Then for any  $\tau \in X(k-2)$

$$\sum_{v \in X_\tau(0)} \phi_\tau(v) = \sum_{v \in X_\tau(0)} \phi(v\tau) = d_{k-2}^* \phi(\tau) = 0.$$

Therefore, by the variational characterization of  $\lambda_2(X_\tau)$

$$\|d_0\phi_\tau\|_{X_\tau}^2 = (d_0^* d_0 \phi_\tau, \phi_\tau)_{X_\tau} \geq \lambda_2(X_\tau) \|\phi_\tau\|_{X_\tau}^2 \geq \lambda(X) \|\phi_\tau\|_{X_\tau}^2. \tag{11}$$

Substituting (11) in (8) we obtain

$$\begin{aligned}
\|d_{k-1}\phi\|_X^2 &= \sum_{\tau \in X(k-2)} \|d_0\phi_\tau\|_{X_\tau}^2 - m(k-1) \|\phi\|_X^2 \\
&\geq \lambda(X) \sum_{\tau \in X(k-2)} \|\phi_\tau\|_{X_\tau}^2 - m(k-1) \|\phi\|_X^2 \\
&= (\lambda(X)k - m(k-1)) \|\phi\|_X^2.
\end{aligned}$$

□

### 3 Laplacians Spectra on $Y_{G,k}$

In this section we prove the following characterization of the spectral gap of complexes that contain the full  $(k-1)$ -skeleton of balanced complexes.

**Proposition 3.1.** *For any subcomplex  $Y_{G,k}^{(k-1)} \subset Y \subset Y_{G,k}$*

$$\mu_{k-1}(Y) = \min \left\{ \frac{\|d_{k-1}\phi\|_Y^2}{\|\phi\|_Y^2} : 0 \neq \phi \in \ker d_{k-2}^* \right\}.$$

We first record some facts concerning the Laplacian spectra of  $Y_{G,k}$ . We will use the notation introduced in Section 2 with Laplacians  $L_j = L_j(Y_{G,k})$ .

**Proposition 3.2.**

(i) For  $0 \leq j \leq k$

$$\begin{aligned} \mathcal{S}(C^j(Y_{G,k}), L_j) &= \{tn : k-j \leq t \leq k+1\}, \\ s(C^j(Y_{G,k}), L_j, tn) &= \binom{k+1}{t} \binom{t}{k-j} \cdot (n-1)^{k+1-t}. \end{aligned} \quad (12)$$

(ii) For  $0 \leq j \leq k$

$$\begin{aligned} \mathcal{S}(\operatorname{Im} d_{j-1}, L_j^-) &= \{tn : k-j+1 \leq t \leq k+1\}, \\ s(\operatorname{Im} d_{j-1}, L_j^-, tn) &= \binom{k+1}{t} \binom{t-1}{k-j} \cdot (n-1)^{k+1-t}. \end{aligned} \quad (13)$$

For  $0 \leq j \leq k-1$

$$\begin{aligned} \mathcal{S}(\operatorname{Im} d_j^*, L_j^+) &= \{tn : k-j \leq t \leq k+1\}, \\ s(\operatorname{Im} d_j^*, L_j^+, tn) &= \binom{k+1}{t} \binom{t-1}{k-j-1} \cdot (n-1)^{k+1-t}. \end{aligned} \quad (14)$$

**Proof.** (i) Recall that  $V_i$  is the  $n$  point space  $\{i\} \times G$ . For  $0 \leq j \leq k$  let

$$E_{k,j} = \left\{ \underline{\epsilon} = (\epsilon_1, \dots, \epsilon_{k+1}) \in \{-1, 0\}^{k+1} : \epsilon_1 + \dots + \epsilon_{k+1} = j - k \right\}.$$

The formula for the spectra of the Laplacians of joins (see e.g. Section 4 in [6]) implies that for  $0 \leq j \leq k$

$$\begin{aligned} \tilde{\mathcal{S}}(C^j(Y_{G,k}), L_j) &= \tilde{\mathcal{S}}(C^j(V_1 * \dots * V_{k+1}), L_j) \\ &= \bigcup_{\underline{\epsilon}=(\epsilon_1, \dots, \epsilon_{k+1}) \in E_{k,j}} \left( \tilde{\mathcal{S}}(C^{\epsilon_1}(V_1), L_{\epsilon_1}) + \dots + \tilde{\mathcal{S}}(C^{\epsilon_{k+1}}(V_{k+1}), L_{\epsilon_{k+1}}) \right). \end{aligned} \quad (15)$$

As  $L_{-1}(V_i)$  is multiplication by  $n$  and  $L_0(V_i)$  is the all ones  $n \times n$  matrix, it follows that  $\mathcal{S}(C^{-1}(V_i), L_{-1}) = \{n\}$  and  $\mathcal{S}(C^0(V_i), L_0) = \{0, n\}$  where  $s(C^0(V_i), L_0, 0) = n-1$  and

$s(C^0(V_i), L_0, n) = 1$ . Fix an  $\underline{\epsilon} = (\epsilon_1, \dots, \epsilon_{k+1}) \in E_{k,j}$ . Then  $I = \{1 \leq i \leq k+1 : \epsilon_i = -1\}$  satisfies  $|I| = k-j$ . The multiset corresponding to  $\underline{\epsilon}$  in (15) is therefore

$$M_{\underline{\epsilon}} = \overbrace{\{n\} + \dots + \{n\}}^{k-j} + \overbrace{\{0, \dots, 0, n\} + \dots + \{0, \dots, 0, n\}}^{j+1}.$$

Clearly  $M_{\underline{\epsilon}}$  consists of the elements  $\{tn : k-j \leq t \leq k+1\}$ , where the multiplicity of  $tn$  is  $\binom{j+1}{t-(k-j)}(n-1)^{k+1-t}$ . Therefore

$$\begin{aligned} s(C^j(Y_{G,k}), L_j, tn) &= |E_{k,j}| \cdot \binom{j+1}{t-(k-j)} (n-1)^{k+1-t} \\ &= \binom{k+1}{k-j} \binom{j+1}{t-(k-j)} (n-1)^{k+1-t} = \binom{k+1}{t} \binom{t}{k-j} \cdot (n-1)^{k+1-t}. \end{aligned}$$

(ii) We argue by decreasing induction on  $j$ . For the base case  $j = k$ , first note that (4) implies that  $0 \notin \mathcal{S}(\text{Im } d_{k-1}, L_k^-)$ . Moreover, as  $L_k^+ = 0$  it follows by (6) that for  $\lambda \neq 0$

$$s(\text{Im } d_{k-1}, L_k^-, \lambda) = s(C^k(Y_{G,k}), L_k, \lambda).$$

Thus, by (12)

$$\mathcal{S}(\text{Im } d_{k-1}, L_k^-) = \mathcal{S}(C^k(Y_{G,k}), L_k) \setminus \{0\} = \{tn : 1 \leq t \leq k+1\}$$

and

$$s(\text{Im } d_{k-1}, L_k^-, tn) = s(C^k(Y_{G,k}), L_k, tn) = \binom{k+1}{t} \cdot (n-1)^{k+1-t}.$$

For the induction step, let  $1 \leq j_0 \leq k-1$  and assume that (13) holds for all  $j_0 < j' \leq k$  and that (14) holds for all  $j_0 < j' \leq k-1$ . Then by (7)

$$\begin{aligned} \mathcal{S}(\text{Im } d_{j_0}^*, L_{j_0}^+) &= \mathcal{S}(\text{Im } d_{j_0}, L_{j_0+1}^-) \\ &= \{tn : k - (j_0 + 1) + 1 \leq t \leq k+1\} = \{tn : k - j_0 \leq t \leq k+1\} \end{aligned}$$

and

$$\begin{aligned} s(\text{Im } d_{j_0}^*, L_{j_0}^+, tn) &= s(\text{Im } d_{j_0}, L_{j_0+1}^-, tn) \\ &= \binom{k+1}{t} \binom{t-1}{k-(j_0+1)} \cdot (n-1)^{k+1-t} = \binom{k+1}{t} \binom{t-1}{k-j_0-1} \cdot (n-1)^{k+1-t}. \end{aligned}$$

Thus (14) holds for  $j = j_0$ . Furthermore, by (6)

$$\begin{aligned} \{tn : k - j_0 \leq t \leq k+1\} &= \mathcal{S}(C^{j_0}(X), L_{j_0}) \\ &= \mathcal{S}(\text{Im } d_{j_0-1}, L_{j_0}^-) \cup \mathcal{S}(\text{Im } d_{j_0}^*, L_{j_0}^+) \end{aligned}$$



and for all  $k - j_0 \leq t \leq k + 1$

$$\begin{aligned}
s \left( \text{Im } d_{j_0-1}, L_{j_0}^-, tn \right) &= s \left( C^{j_0}(Y_{G,k}), L_{j_0}, tn \right) - s \left( \text{Im } d_{j_0}^*, L_{j_0}^+, tn \right) \\
&= \binom{k+1}{t} \binom{t}{k-j_0} \cdot (n-1)^{k+1-t} - \binom{k+1}{t} \binom{t-1}{k-j_0-1} \cdot (n-1)^{k+1-t} \\
&= \binom{k+1}{t} \binom{t-1}{k-j_0} \cdot (n-1)^{k+1-t} \\
&= \begin{cases} \binom{k+1}{t} \binom{t-1}{k-j_0} \cdot (n-1)^{k+1-t} & k-j_0+1 \leq t \leq k+1, \\ 0 & t = k-j_0. \end{cases}
\end{aligned}$$

Thus (13) holds for  $j = j_0$ , thereby completing the inductive proof of (ii).  $\square$

**Proof of Proposition 3.1.** Let  $Y_{G,k}^{(k-1)} \subset Y \subset Y_{G,k}$ . Note that if  $\phi \in C^{k-1}(Y) = C^{k-1}(Y_{A,k})$  then  $\|\phi\|_Y = \|\phi\|_{Y_{G,k}}$ ,  $\|d_{k-1}\phi\|_Y \leq \|d_{k-1}\phi\|_{Y_{G,k}}$  and  $\|d_{k-2}^*\phi\|_Y = \|d_{k-2}^*\phi\|_{Y_{G,k}}$ . The cases  $j = k - 1$  of (14) and (13) then imply respectively that

$$\begin{aligned}
\alpha_+ &:= \min \left\{ \frac{\|d_{k-1}\phi\|_Y^2}{\|\phi\|_Y^2} : 0 \neq \phi \in \ker d_{k-2}^* \right\} \\
&\leq \min \left\{ \frac{\|d_{k-1}\phi\|_{Y_{G,k}}^2}{\|\phi\|_{Y_{G,k}}^2} : 0 \neq \phi \in \ker d_{k-2}^* \right\} \\
&= \min \left\{ \frac{\|d_{k-1}\phi\|_{Y_{G,k}}^2}{\|\phi\|_{Y_{G,k}}^2} : 0 \neq \phi \in d_{k-1}^* \left( C^k(Y_{G,k}) \right) \right\} \\
&= \mu_{k-1}^+(Y_{G,k}) = \min\{tn : 1 \leq t \leq k+1\} = n
\end{aligned}$$

and

$$\begin{aligned}
\alpha_- &:= \min \left\{ \frac{\|d_{k-2}^*\phi\|_Y^2}{\|\phi\|_Y^2} : 0 \neq \phi \in \text{Im } d_{k-2} \right\} \\
&= \min \left\{ \frac{\|d_{k-2}^*\phi\|_{Y_{G,k}}^2}{\|\phi\|_{Y_{G,k}}^2} : 0 \neq \phi \in \text{Im } d_{k-2} \right\} \\
&= \mu_{k-1}^-(Y_{G,k}) = \min\{tn : 2 \leq t \leq k+1\} = 2n.
\end{aligned}$$

Therefore  $\alpha_+ < \alpha_-$ . Moreover,  $H^{k-1}(Y_{G,k}) = 0$  together with (3) and (5) imply that there is an orthogonal decomposition

$$C^{k-1}(Y) = C^{k-1}(Y_{G,k}) = \text{Im } d_{k-2} \oplus \ker d_{k-2}^*.$$

Let  $P_1, P_2$  denote the orthogonal projections of  $C^{k-1}(Y)$  onto  $\text{Im } d_{k-2}$  and  $\ker d_{k-2}^*$  respec-

tively. Then

$$\begin{aligned}
\mu_{k-1}(Y) &= \min \left\{ \frac{(L_{k-1}\phi, \phi)_Y}{(\phi, \phi)_Y} : 0 \neq \phi \in C^{k-1}(Y) \right\} \\
&= \min \left\{ \frac{\|d_{k-2}^*\phi\|_Y^2 + \|d_{k-1}\phi\|_Y^2}{\|\phi\|_Y^2} : 0 \neq \phi \in C^{k-1}(Y) \right\} \\
&= \min \left\{ \frac{\|d_{k-2}^*P_1\phi\|_Y^2 + \|d_{k-1}P_2\phi\|_Y^2}{\|P_1\phi\|_Y^2 + \|P_2\phi\|_Y^2} : 0 \neq \phi \in C^{k-1}(Y) \right\} \\
&= \min \left\{ \frac{\|d_{k-2}^*\phi_1\|_Y^2 + \|d_{k-1}\phi_2\|_Y^2}{\|\phi_1\|_Y^2 + \|\phi_2\|_Y^2} : (0, 0) \neq (\phi_1, \phi_2) \in \text{Im } d_{k-2} \times \ker d_{k-2}^* \right\} \\
&= \min \left\{ \frac{\alpha_- \|\phi_1\|_Y^2 + \alpha_+ \|\phi_2\|_Y^2}{\|\phi_1\|_Y^2 + \|\phi_2\|_Y^2} : (0, 0) \neq (\phi_1, \phi_2) \in \text{Im } d_{k-2} \times \ker d_{k-2}^* \right\} \\
&= \min\{\alpha_-, \alpha_+\} = \alpha_+.
\end{aligned}$$

□

## 4 The Fourier Transform

Let  $\mathcal{L}(G)$  denote the algebra of complex valued functions on  $G$  with the convolution product  $\phi * \psi(x) = \sum_{y \in G} \phi(y)\psi(y^{-1}x)$ . The inner product on  $\mathcal{L}(G)$  is given by

$$\langle \phi, \psi \rangle = \sum_{x \in G} \phi(x) \overline{\psi(x)}.$$

The Frobenius inner product and norm on  $M_d(\mathbb{C})$  are given respectively by  $\langle S, T \rangle = \text{tr}(ST^*)$  and  $\|T\|_F = \sqrt{\langle T, T \rangle} = \sqrt{\text{tr}(TT^*)}$ . The Frobenius norm of a product satisfies

$$\|ST\|_F \leq \|S\| \cdot \|T\|_F. \quad (16)$$

Let  $\mathcal{R}(G)$  denote the algebra  $\prod_{\rho \in \widehat{G}} M_{d_\rho}(\mathbb{C})$  with coordinate wise addition and multiplication. Define an inner product on  $\mathcal{R}(G)$  by

$$\left\langle (S_\rho : \rho \in \widehat{G}), (T_\rho : \rho \in \widehat{G}) \right\rangle = \frac{1}{n} \sum_{\rho} d_\rho \langle S_\rho, T_\rho \rangle = \frac{1}{n} \sum_{\rho} d_\rho \text{tr}(S_\rho T_\rho^*).$$

The associated norm is given by

$$\left\| (T_\rho : \rho \in \widehat{G}) \right\|_F = \left( \frac{1}{n} \sum_{\rho \in \widehat{G}} d_\rho \|T_\rho\|_F^2 \right)^{\frac{1}{2}}.$$

**Definition 4.1.** For  $\phi \in \mathcal{L}(G)$  and a representation  $\rho$  of  $G$  of degree  $d$  let

$$\widehat{\phi}(\rho) = \sum_{x \in G} \phi(x) \rho(x) \in M_d(\mathbb{C}).$$

The **Fourier Transform**  $\mathcal{F} : \mathcal{L}(G) \rightarrow \mathcal{R}(G)$  is given by

$$\mathcal{F}(\phi) = \left( \widehat{\phi}(\rho) : \rho \in \widehat{G} \right).$$

A basic result in representation theory (see e.g. exercise 3.32 in [7]) asserts that  $\mathcal{F}$  is an isomorphism of algebras and an isometry. In particular,  $\mathcal{F}$  satisfies the Parseval identity: For any  $\phi, \psi \in \mathcal{L}(G)$

$$\langle \phi, \psi \rangle = \langle \mathcal{F}(\phi), \mathcal{F}(\psi) \rangle = \frac{1}{n} \sum_{\rho \in \widehat{G}} d_\rho \langle \widehat{\phi}(\rho), \widehat{\psi}(\rho) \rangle. \quad (17)$$

## 5 The $(k-1)$ -Spectral Gap of $Y_{A,k}$

In this section we prove Theorem 1.5. Let  $X = Y_{A,k}$ . We need two preliminary observations. Let  $\mathcal{C}_A = Y_{A,1}$ , i.e. the graph on the vertex set  $V(\mathcal{C}_A) = \{1, 2\} \times G$  with edge set

$$E(\mathcal{C}_A) = \{(1, x_1), (2, x_2) : x_1 \cdot x_2 \in A\}.$$

**Claim 5.1.** *For any  $\tau \in X(k-2)$ , the graph  $X_\tau = \text{lk}(X, \tau)$  is isomorphic to  $\mathcal{C}_A$ .*

**Proof.** Let  $\tau = \{(j, y_j)\}_{j \in J}$  where  $J \subset [k+1] := \{1, \dots, k+1\}$  and  $|J| = k-1$ . Let  $[k+1] \setminus J = \{i_1 < i_2\}$ . Let  $z_1 = y_1 \cdots y_{i_1-1}$ ,  $z_2 = y_{i_1+1} \cdots y_{i_2-1}$  and  $z_3 = y_{i_2+1} \cdots y_{k+1}$ . Then  $X_\tau$  is the graph on the vertex set  $V_\tau = \{i_1, i_2\} \times G$  with edge set

$$E_\tau = \{(i_1, x_{i_1}), (i_2, x_{i_2}) : z_1 x_{i_1} z_2 x_{i_2} z_3 \in A\}.$$

Let  $\varphi : V_\tau \rightarrow V(\mathcal{C}_A)$  be given by

$$\varphi((i_t, x_{i_t})) = \begin{cases} (1, z_1 x_{i_1} z_2) & t = 1, \\ (2, x_{i_2} z_3) & t = 2. \end{cases}$$

Then  $\varphi$  is an isomorphism between  $X_\tau$  and  $\mathcal{C}_A$ . □

The next result gives a lower bound on the spectral gap of  $\mathcal{C}_A$ .

**Proposition 5.2.**

$$\lambda_2(\mathcal{C}_A) \geq |A| - \nu(A).$$

**Proof.** Let  $\phi \in C^0(V(\mathcal{C}_A))$  such that  $\sum_{v \in V(\mathcal{C}_A)} \phi(v) = 0$ . For  $i = 1, 2$  let  $\phi_i \in \mathcal{L}(G)$  be given by  $\phi_i(x) = \phi((i, x))$ . Define  $\psi \in \mathcal{L}(G)$  by  $\psi(x) = \phi_2(x^{-1})$  and for  $a \in A$  let  $\psi_a(x) = \psi(a^{-1}x) = \phi_2(x^{-1}a)$ . Then

$$\begin{aligned} \widehat{\phi}_1(\mathbf{1}) + \widehat{\psi}_a(\mathbf{1}) &= \left( \sum_{x \in G} \phi_1(x) \right) + \left( \sum_{x \in G} \psi_a(x) \right) \\ &= \left( \sum_{x \in G} \phi_1(x) \right) + \left( \sum_{x \in G} \phi_2(x) \right) = \sum_{v \in V(\mathcal{C}_A)} \phi(v) = 0. \end{aligned}$$

Hence

$$\widehat{\phi}_1(\mathbf{1}) \cdot \widehat{\psi}_a(\mathbf{1}) = -\widehat{\phi}_1(\mathbf{1})^2 \leq 0. \quad (18)$$

For any  $\rho \in \widehat{G}$

$$\begin{aligned}
\widehat{\psi}_a(\rho) &= \sum_{x \in G} \psi_a(x) \rho(x) = \sum_{x \in G} \phi_2(x^{-1}a) \rho(x) \\
&= \sum_{y \in G} \phi_2(y) \rho(ay^{-1}) = \rho(a) \sum_{y \in G} \phi_2(y) \rho(y^{-1}) = \rho(a) \widehat{\psi}(\rho).
\end{aligned} \tag{19}$$

Using the Parseval identity (17) together with (18), (19) and (16) we obtain

$$\begin{aligned}
&\sum_{a \in A} \langle \phi_1, \psi_a \rangle = \sum_{a \in A} \langle \mathcal{F}(\phi_1), \mathcal{F}(\psi_a) \rangle \\
&= \frac{1}{n} \sum_{a \in A} \left( \sum_{\rho \in \widehat{G}} d_\rho \langle \widehat{\phi}_1(\rho), \widehat{\psi}_a(\rho) \rangle \right) \\
&= \frac{1}{n} \sum_{a \in A} \left( \widehat{\phi}_1(\mathbf{1}) \cdot \widehat{\psi}_a(\mathbf{1}) + \sum_{\rho \in \widehat{G}_+} d_\rho \langle \widehat{\phi}_1(\rho), \widehat{\psi}_a(\rho) \rangle \right) \\
&= -\frac{|A|}{n} \widehat{\phi}_1(\mathbf{1})^2 + \frac{1}{n} \sum_{a \in A} \sum_{\rho \in \widehat{G}_+} d_\rho \langle \widehat{\phi}_1(\rho), \rho(a) \widehat{\psi}(\rho) \rangle \\
&= -\frac{|A|}{n} \widehat{\phi}_1(\mathbf{1})^2 + \frac{1}{n} \sum_{\rho \in \widehat{G}_+} d_\rho \langle \widehat{\phi}_1(\rho), \widehat{1}_A(\rho) \widehat{\psi}(\rho) \rangle \\
&\leq \frac{1}{n} \sum_{\rho \in \widehat{G}_+} d_\rho \|\widehat{\phi}_1(\rho)\|_F \cdot \|\widehat{1}_A(\rho) \cdot \widehat{\psi}(\rho)\|_F \\
&\leq \frac{1}{n} \sum_{\rho \in \widehat{G}_+} d_\rho \|\widehat{\phi}_1(\rho)\|_F \cdot \|\widehat{1}_A(\rho)\| \cdot \|\widehat{\psi}(\rho)\|_F \\
&\leq \nu(A) \sum_{\rho \in \widehat{G}} \left( \sqrt{\frac{d_\rho}{n}} \|\widehat{\phi}_1(\rho)\|_F \right) \cdot \left( \sqrt{\frac{d_\rho}{n}} \|\widehat{\psi}(\rho)\|_F \right) \\
&\leq \nu(A) \left( \frac{1}{n} \sum_{\rho \in \widehat{G}} d_\rho \|\widehat{\phi}_1(\rho)\|_F^2 \right)^{\frac{1}{2}} \left( \frac{1}{n} \sum_{\rho \in \widehat{G}} d_\rho \|\widehat{\psi}(\rho)\|_F^2 \right)^{\frac{1}{2}} \\
&= \nu(A) \cdot \|\phi_1\| \cdot \|\psi\| = \nu(A) \cdot \|\phi_1\| \cdot \|\phi_2\|.
\end{aligned} \tag{20}$$

Finally by (20)

$$\begin{aligned}
\|d_0\phi\|_{\mathcal{C}_A}^2 &= \sum_{\{(x_1, x_2) \in G^2 : x_1 x_2 \in A\}} d_0\phi([(1, x_1), (2, x_2)])^2 \\
&= \sum_{\{(x_1, x_2) \in G^2 : x_1 x_2 \in A\}} (\phi_2(x_2) - \phi_1(x_1))^2 \\
&= \sum_{a \in A} \sum_{x \in G} (\phi_2(x^{-1}a) - \phi_1(x))^2 \\
&= \sum_{a \in A} \sum_{x \in G} (\phi_1(x)^2 + \phi_2(x^{-1}a)^2 - 2\phi_1(x)\psi_a(x)) \\
&= |A| (\|\phi_1\|^2 + \|\phi_2\|^2) - 2 \sum_{a \in A} \langle \phi_1, \psi_a \rangle \\
&\geq |A| \cdot \|\phi\|_{\mathcal{C}_A}^2 - 2\nu(A) \cdot \|\phi_1\| \cdot \|\phi_2\| \\
&\geq |A| \cdot \|\phi\|_{\mathcal{C}_A}^2 - \nu(A) (\|\phi_1\|^2 + \|\phi_2\|^2) \\
&= (|A| - \nu(A)) \|\phi\|_{\mathcal{C}_A}^2.
\end{aligned}$$

Therefore  $\lambda_2(\mathcal{C}_A) \geq |A| - \nu(A)$ . □

**Proof of Theorem 1.5.** Clearly  $\deg_X(\sigma) = |A|$  for all  $\sigma \in X(k-1)$ . By Claim 5.1 and Proposition 5.2

$$\lambda(X) = \min \{\lambda_2(X_\tau) : \tau \in X(k-2)\} = \lambda_2(\mathcal{C}_A) \geq |A| - \nu(A). \quad (21)$$

Using Proposition 3.1, Garland's Theorem 2.3 and (21) it follows that

$$\begin{aligned}
\mu_{k-1}(X) &= \min \left\{ \frac{\|d_{k-1}\phi\|_X^2}{\|\phi\|_X^2} : 0 \neq \phi \in \ker d_{k-2}^* \right\} \\
&\geq k\lambda(X) - (k-1)|A| \geq k(|A| - \nu(A)) - (k-1)|A| \\
&= |A| - k \cdot \nu(A).
\end{aligned}$$

□

## 6 The Spectral Gap of a Random $Y_{A,k}$

In this section we prove Theorem 1.6. We will use the following matrix version of Bernstein's large deviation inequality due to Tropp (Theorem 1.6 in [16]).

**Theorem 6.1** ([16]). *Let  $\{X_i\}_{i=1}^m$  be independent random variables taking values in  $M_d(\mathbb{C})$  such that  $E[X_i] = 0$  and  $\|X_i\| \leq R$  for all  $1 \leq i \leq m$ , and let*

$$\sigma^2 = \max \left\{ \left\| \sum_{i=1}^m E[X_i X_i^*] \right\|, \left\| \sum_{i=1}^m E[X_i^* X_i] \right\| \right\}.$$

Then for any  $\lambda \geq 0$

$$\Pr \left[ \left\| \sum_{i=1}^m X_i \right\| \geq \lambda \right] \leq 2d \exp \left( -\frac{3\lambda^2}{6\sigma^2 + 2R\lambda} \right).$$

**Proof of Theorem 1.6.** Let  $k \geq 1$  and  $0 < \epsilon < 1$  be fixed and let  $m$  be an integer such that  $\frac{9k^2 \log D(G)}{\epsilon^2} \leq m \leq \sqrt{n}$ . Let  $\Omega$  denote the uniform probability space of all  $m$ -subsets of  $G$ . Suppose that  $A \in \Omega$  satisfies  $\nu(A) \leq \epsilon k^{-1}m$ . Then by Theorem 1.5

$$\begin{aligned} \mu_{k-1}(X_{A,k}) &\geq |A| - k \cdot \nu(A) \\ &\geq m - k \cdot \epsilon k^{-1}m = (1 - \epsilon)m. \end{aligned}$$

Theorem 1.6 will therefore follow from

**Proposition 6.2.**

$$\Pr_{\Omega} \left[ A \in \Omega : \max_{\rho \in \widehat{G}_+} \|\widehat{1}_A(\rho)\| > \epsilon k^{-1}m \right] < \frac{6}{n}.$$

**Proof.** Let  $\rho \in \widehat{G}_+$  be fixed and let  $\lambda = \epsilon k^{-1}m$ . Let  $\Omega'$  denote the uniform probability space  $G^m$ , and for  $1 \leq i \leq m$  let  $X_i$  be the random variable defined on  $\omega' = (a_1, \dots, a_m) \in \Omega'$  by  $X_i(\omega') = \rho(a_i) \in U(d_\rho)$ . As  $\rho \in \widehat{G}_+$ , it follows by Schur's Lemma that  $E[X_i] = 0$ . It is straightforward to check the  $X_i$ 's also satisfy the additional conditions of Theorem 6.1 with  $\sigma^2 = m$  and  $R = 1$ . Hence

$$\begin{aligned} \Pr_{\Omega'} \left[ \omega' \in \Omega' : \left\| \sum_{i=1}^m X_i(\omega') \right\| \geq \lambda \right] &\leq 2d_\rho \exp \left( -\frac{3\lambda^2}{6\sigma^2 + 2R\lambda} \right) \\ &= 2d_\rho \exp \left( -\frac{3(\epsilon k^{-1}m)^2}{6m + 2\epsilon k^{-1}m} \right) < 2d_\rho \exp \left( -\frac{\epsilon^2 m}{3k^2} \right) \\ &\leq 2d_\rho \exp(-3 \log D(G)) = 2d_\rho D(G)^{-3}. \end{aligned} \tag{22}$$

Let  $\Omega'' = \{(a_1, \dots, a_m) \in G^m : a_i \neq a_j \text{ for } i \neq j\} \subset \Omega'$  denote the uniform probability space consisting of all sequences in  $G^m$  with distinct elements. The assumption  $m \leq \sqrt{n}$  implies that

$$(\Pr_{\Omega'}[\Omega''])^{-1} = \prod_{i=1}^m \frac{n}{n-i+1} \leq \left( \frac{n}{n-m+1} \right)^m \leq \exp \left( \frac{(m-1)m}{n-m+1} \right) \leq e. \tag{23}$$

Combining (22) and (23) we obtain

$$\begin{aligned} &\Pr_{\Omega} \left[ A \in \Omega : \|\widehat{1}_A(\rho)\| \geq \epsilon k^{-1}m \right] \\ &= \Pr_{\Omega''} \left[ \omega'' \in \Omega'' : \left\| \sum_{i=1}^m X_i(\omega'') \right\| \geq \epsilon k^{-1}m \right] \\ &\leq \Pr_{\Omega'} \left[ \omega' \in \Omega' : \left\| \sum_{i=1}^m X_i(\omega') \right\| \geq \epsilon k^{-1}m \right] \cdot (\Pr_{\Omega'}[\Omega''])^{-1} \\ &< 6d_\rho D(G)^{-3}. \end{aligned} \tag{24}$$

Note that

$$D(G)^2 = \left( \sum_{\rho \in \widehat{G}} d_\rho \right)^2 \geq \sum_{\rho \in \widehat{G}} d_\rho^2 = n.$$

Using the union bound and (24) it thus follows that

$$\Pr_{\Omega} \left[ \nu(A) \geq \epsilon k^{-1} m \right] < 6 \sum_{\rho \in \hat{G}} d_{\rho} D(G)^{-3} = 6D(G)^{-2} \leq \frac{6}{n}.$$

□

## 7 $Y_{A,k}$ for a Subgroup $A$

In this section we prove Theorem 1.8. Our main tool will be the Wedge Lemma of Ziegler and Živaljević (Lemma 1.8 in [17]). The version below appears in [9]. For a poset  $(P, \prec)$  and  $p \in P$  let  $P_{\prec p} = \{q \in P : q \prec p\}$ . Let  $\Delta(P)$  denote the order complex of  $P$ . Let  $Y$  be a regular CW-complex and let  $\{Z_i\}_{i=1}^{\ell}$  be subcomplexes of  $Y$  such that  $\bigcup_{i=1}^{\ell} Z_i = Y$ . Let  $(P, \prec)$  be the poset whose elements index all distinct partial intersections  $\bigcap_{j \in J} Z_j$ , where  $\emptyset \neq J \subset [\ell]$ . Let  $U_p$  denote the partial intersection indexed by  $p \in P$ , and let  $\prec$  denote reverse inclusion, i.e.  $p \prec q$  if  $U_q \subsetneq U_p$ . Note  $\{\emptyset\} * U_p = U_p$ .

**Wedge Lemma** ([17, 9]). *Suppose that for any  $p \in P$  there exists a  $c_p \in U_p$  such that the inclusion  $\bigcup_{q \succ p} U_q \hookrightarrow U_p$  is homotopic to the constant map to  $c_p$ . Then*

$$Y \simeq \bigvee_{p \in P} \Delta(P_{\prec p}) * U_p.$$

**Proof of Theorem 1.8.** Let  $A$  be a subgroup of  $G$  of size  $|A| = m$  and let  $\ell = \frac{n}{m}$ . Let  $g_1, \dots, g_{\ell} \in G$  be coset representatives of  $A$ , i.e.  $G = \bigcup_{i=1}^{\ell} g_i A$ . (i) The graph  $Y_{A,1}$  is isomorphic to the disjoint union  $\coprod_{i=1}^{\ell} A g_i^{-1} * g_i A$ . This implies (1) since each  $A g_i^{-1} * g_i A$  is a complete  $m$  by  $m$  bipartite graph and hence homotopic to a wedge of  $(m-1)^2$  circles. (ii) Let  $k \geq 2$ . For  $1 \leq i \leq \ell$  let  $W_{k,i} = \{k+1\} \times g_i A \subset V_{k+1}$  and let

$$Z_{k,i} = Y_{A g_i^{-1}, k-1} * W_{k,i} \cong Y_{A, k-1} * [m]. \quad (25)$$

Then  $\bigcup_{i=1}^{\ell} Z_{k,i} = Y_{A,k}$ . Indeed, let  $x_1, \dots, x_{k+1} \in G$ , and suppose that  $x_{k+1} \in g_i A$ . Then

$$\begin{aligned} \sigma = \{(1, x_1), \dots, (k, x_k), (k+1, x_{k+1})\} \in Y_{A,k} &\iff x_1 \cdots x_{k+1} \in A \\ &\iff x_1 \cdots x_k \in A g_i^{-1} \iff \sigma \in Z_{k,i}(k). \end{aligned}$$

Moreover, for any  $1 \leq j \neq j' \leq \ell$

$$Z_{k,j} \cap Z_{k,j'} = \bigcap_{i=1}^{\ell} Z_{k,i} = Y_{G, k-1}^{(k-2)}. \quad (26)$$

The reduced Euler characteristic of  $Y_{G, k-1}$  satisfies  $\tilde{\chi}(Y_{G, k-1}) = (-1)^{k-1} (n-1)^k$ . Hence

$$N_k := (-1)^{k-2} \tilde{\chi} \left( Y_{G, k-1}^{(k-2)} \right) = (-1)^{k-2} \left( \tilde{\chi}(Y_{G, k-1}) - (-1)^{k-1} n^k \right) = n^k - (n-1)^k.$$

As  $Y_{G,k-1}^{(k-2)}$  is a matroidal complex of rank  $k-1$ , it follows (see e.g. Theorem 7.8.1 in [4]) that

$$Y_{G,k-1}^{(k-2)} \simeq \bigvee^{N_k} S^{k-2}. \quad (27)$$

Eq. (26) implies that the intersection poset  $(P, \prec)$  of the cover  $\{Z_{k,i}\}_{i=1}^\ell$  is  $P = [\ell] \cup \{\widehat{1}\}$ , where  $i \in [\ell]$  represents  $Z_{k,i}$ ,  $\widehat{1}$  represents  $Y_{G,k-1}^{(k-2)}$ ,  $[\ell]$  is an antichain and  $i \prec \widehat{1}$  for all  $i \in [\ell]$ . Note that  $\Delta(P_{\prec i}) = \emptyset$  for all  $i \in [m]$  and  $\Delta(P_{\prec \widehat{1}})$  is the discrete space  $[\ell]$ . We proceed to prove (2) by induction on  $k$ . We first establish the induction step. Let  $k \geq 3$  and assume that (2) holds for  $k-1$ . Then  $Z_{k,i} \cong Y_{A,k-1} * [m]$  is homotopy equivalent to a wedge of spheres of dimensions  $k-1$  and  $k$ . As  $Y_{G,k-1}^{(k-2)}$  is a wedge of  $(k-2)$ -spheres, it follows that the inclusion  $Y_{G,k-1}^{(k-2)} \hookrightarrow Z_{k,i}$  is null homotopic. Applying the Wedge Lemma together with (25), (27) and the induction hypothesis, we obtain

$$\begin{aligned} Y_{A,k} &\simeq \left( \bigvee_{i \in [\ell]} \Delta(P_{\prec i}) * Z_{k,i} \right) \vee \left( \Delta(P_{\prec \widehat{1}}) * Y_{G,k-1}^{(k-2)} \right) \\ &= \left( \bigvee_{i \in [\ell]} Z_{k,i} \right) \vee \left( [\ell] * Y_{G,k-1}^{(k-2)} \right) \\ &\cong \left( \bigvee_{i \in [\ell]} Y_{A,k-1} * [m] \right) \vee \left( [\ell] * \bigvee^{N_k} S^{k-2} \right) \\ &\simeq \left( \bigvee_{i \in [\ell]} \left( \bigvee^{\gamma_0(m,k-1)} S^{k-2} \vee \bigvee^{\gamma_1(m,k-1)} S^{k-1} \right) * [m] \right) \vee \left( [\ell] * \bigvee^{N_k} S^{k-2} \right) \\ &\simeq \bigvee_{i \in [\ell]} \left( \bigvee^{(m-1)\gamma_0(m,k-1)} S^{k-1} \vee \bigvee^{(m-1)\gamma_1(m,k-1)} S^k \right) \vee \left( \bigvee^{(\ell-1)N_k} S^{k-1} \right) \\ &\simeq \bigvee^{t_0} S^{k-1} \vee \bigvee^{t_1} S^k, \end{aligned} \quad (28)$$

where

$$\begin{aligned} t_0 &= \ell(m-1)\gamma_0(m,k-1) + (\ell-1)N_k \\ &= \ell(m-1) \left( (n-m)n^{k-1} + \ell^{k-1}(m-1)^k - (n-1)^k \right) + (\ell-1) \left( n^k - (n-1)^k \right) \\ &= (n-m)n^k + \ell^k(m-1)^{k+1} - (n-1)^{k+1} = \gamma_0(m,k) \end{aligned}$$

and

$$\begin{aligned} t_1 &= \ell(m-1)\gamma_1(m,k-1) = \ell(m-1) \left( \ell^{k-1}(m-1)^k \right) \\ &= \ell^k(m-1)^{k+1} = \gamma_1(m,k). \end{aligned}$$

This completes the induction step. To prove (2) for  $k=2$ , first note that assumptions of the Wedge lemma hold for the decomposition  $Y_{A,2} = \bigcup_{i=1}^\ell Z_{2,i}$ . Arguing as in (28), it thus



follows that

$$\begin{aligned}
Y_{A,2} &\simeq \left( \bigvee_{i \in [\ell]} Y_{A,1} * [m] \right) \vee \left( [\ell] * \bigvee^{N_2} S^0 \right) \\
&\simeq \left( \bigvee_{i \in [\ell]} \left( \prod_{j=1}^{\ell} \bigvee^{(m-1)^2} S^1 \right) * [m] \right) \vee \left( [\ell] * \bigvee^{2n-1} S^0 \right) \\
&\simeq \bigvee^{\ell} \bigvee^{m-1} \left( \bigvee^{\ell(m-1)^2} S^2 \vee \bigvee^{\ell-1} S^1 \right) \vee \bigvee^{(\ell-1)(2n-1)} S^1 \\
&= \bigvee^{t_0} S^1 \vee \bigvee^{t_1} S^2,
\end{aligned}$$

where

$$\begin{aligned}
t_0 &= \ell(m-1)(\ell-1) + (\ell-1)(2n-1) \\
&= (n-m)n^2 + \ell^2(m-1)^3 - (n-1)^3 = \gamma_0(m, 2)
\end{aligned}$$

and

$$t_1 = \ell^2(m-1)^3 = \gamma_1(m, k).$$

This completes the proof of the base case  $k = 2$  and of the Theorem. □

## 8 Concluding Remarks

In this paper we studied the  $(k-1)$ -spectral gap of complexes  $Y_{A,k}$  where  $A$  is a subset of a finite group  $G$ . Our main results included a lower bound on  $\mu_{k-1}(Y_{A,k})$  in terms of the Fourier transform of  $1_A$  and a proof that for a sufficiently large constant  $c(k, \epsilon)$ , if  $A$  is a random subset of  $G$  of size at least  $c(k, \epsilon) \log D(G)$ , then  $Y_{A,k}$  has a nearly optimal  $(k-1)$ -th spectral gap. In view of Remark 2.1(ii) it would be interesting to find suitable counterparts of Theorems 1.5 and 1.6 for other robustness measures of cohomological triviality, e.g. for coboundary expansion. We briefly recall the relevant definitions. For a simplicial complex  $X$  and a binary  $k$ -cochain  $\phi \in C^k(X; \mathbb{F}_2)$ , let

$$\|\phi\|_{\text{H}} = |\{\sigma \in X(k) : \phi(\sigma) \neq 0\}|$$

denote the Hamming norm of  $\phi$  and let

$$\|\phi\|_{\text{csy}} = \min \left\{ \|\phi + d_{k-1}\psi\|_{\text{H}} : \psi \in C^{k-1}(X; \mathbb{F}_2) \right\}$$

denote the cosystolic norm of  $\phi$ . The  $k$ -th coboundary expansion constant of  $X$  (see e.g. [13]) is given by

$$h_k(X) = \min \left\{ \frac{\|d_k\phi\|_{\text{H}}}{\|\phi\|_{\text{csy}}} : \phi \in C^k(X; \mathbb{F}_2) \setminus B^k(X; \mathbb{F}_2) \right\}.$$

In light of Theorem 1.6 we suggest the following

**Conjecture 8.1.** For any fixed  $k \geq 1$  there exist constants  $C(k) < \infty$  and  $\epsilon(k) > 0$  such that for any group  $G$ , the random balanced Cayley complexes  $Y_{A,k}$  with  $|A| = C(k) \log D(G)$  satisfy  $h_{k-1}(Y_{A,k}) \geq \epsilon(k)$  a.a.s. as  $|G| \rightarrow \infty$ .

In a different direction, consider the following example of balanced Cayley complexes. Let  $p, q$  be distinct odd primes such that  $q > 2\sqrt{p}$  and  $\left(\frac{p}{q}\right) = 1$ , and let  $G_q = PSL_2(\mathbb{F}_q)$ . The celebrated construction of Ramanujan graphs by Lubotzky, Phillips and Sarnak [14] implies that there exists a subset  $S_{p,q} \subset G_q$  of cardinality  $|S_{p,q}| = p + 1$  such that  $\nu(S_{p,q}) \leq 2\sqrt{p}$ . If  $p \geq 4k^2$  then by Theorem 1.5

$$\mu_{k-1}(Y_{S_{p,q},k}) \geq |S_{p,q}| - k \cdot \nu(S_{p,q}) \geq (p + 1) - 2k\sqrt{p} \geq 1. \quad (29)$$

The following conjecture may be viewed as a coboundary expansion analogue of (29).

**Conjecture 8.2.** For any fixed  $k \geq 1$  there exist constants  $p_0(k) < \infty$  and  $\epsilon_0(k) > 0$  such that if  $p > p_0(k)$ ,  $q > 2\sqrt{p}$  and  $\left(\frac{p}{q}\right) = 1$  then  $h_{k-1}(Y_{S_{p,q},k}) \geq \epsilon_0(k)$ .

### Conflict of Interest Statement

The author states that there is no conflict of interest.

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